

Tri-Bimaximal Neutrino Mixing and the T_{13} Flavor Symmetry

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Abstract

We present a supersymmetric T_{13} model for the tri-bimaximal neutrino mixing, and the complete flavor group is $T_{13} \times Z_4 \times Z_2$. At leading order, the residual symmetry of the charged lepton sector is Z_3 , and the T_{13} symmetry is broken completely in the neutrino sector. The charged lepton mass hierarchies are determined by the spontaneous breaking of the flavor symmetry, both the type I see-saw mechanism and the Weinberg operator contribute to generating the light neutrino masses. Tri-bimaximal mixing is exact at leading order while subleading contributions introduce corrections of order λ_c^2 to the three lepton mixing angles. The vacuum alignment and subleading corrections are studied in detail, a moderate hierarchy of order λ_c between the vacuum expectation values of the flavon fields in the charged lepton and neutrino sectors can be accommodated.

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1 Introduction

The presence of two large and one small mixing angles in the lepton sector [1–3],

$$0.27 < \sin^2 \theta_{12} < 0.37, \quad 0.39 < \sin^2 \theta_{23} < 0.64, \quad \sin^2 \theta_{13} < 0.040 (0.044) \quad \text{at } 3\sigma \quad (1)$$

suggests that the observed neutrino mixing matrix is remarkably compatible with the so called tri-bimaximal (TB) structure [4] within measurement errors. The simple form of the TB mixing matrix implies an underlying family symmetry between the three generations of leptons. It has been realized that the TB mixing can naturally arise as the result of a particular vacuum alignment of scalars that break spontaneously certain flavor symmetries. In the past years, much effort has been devoted to produce TB mixing based on some family symmetry. It has been shown that TB mixing can be understood with the help of discrete flavor symmetries, such as A_4 [5–10], T_7 [11], T' [12], S_4 [13,14] and $\Delta(27)$ [15], or continuous flavor symmetry $SO(3)$ [16] and $SU(3)$ [17]. Discrete non-abelian groups appear to be particularly suitable to reproduce the TB mixing pattern, some higher order discrete groups such as A_5 [18], $\Delta(54)$ [19], $\Sigma(81)$ [20] and $PSL_2(7)$ [21] are also considered for neutrino mixing besides the above mentioned simple groups, the extension of the discrete flavor symmetry to the quark sector and grand unification theory (GUT) have been investigated as well [7,8], please see Refs. [22,23] for a review. In this work, we shall study another 39 element simple discrete group T_{13} in flavor model building, which has gained much less attention.

Recently 76 discrete groups with 3-dimensional representation were scanned, it is suggested that T_{13} is the group with the largest fraction of TB mixing models [24]. But the authors set all the couplings to be equal to 1, the vacuum expectation values are chosen to be 0 or 1, and the vacuum alignment is not considered dynamically in Ref. [24]. It is very interesting to investigate the possible consistent realizations of TB mixing based on T_{13} group from this point of view. As far as we know, the T_{13} group as a discrete flavor symmetry has not been discussed extensively. We note that a T_{13} flavor model was put forward in Ref. [25], and its implication in the indirect detection of dark matter was studied. However, the motivation is not to produce the TB mixing². We have tried many possible assignments for the involved fields, we find the TB mixing can be produced exactly at leading order (LO) in some scenarios, but meanwhile we face the difficulties that the first and third light neutrino are degenerate or the corresponding vacuum alignment is very difficult to be realized or some other problems. In particular, the realizations of TB mixing based on T_{13} symmetry are drastically constrained after taking into account the vacuum alignment issue. After lots of trial and error, we construct a T_{13} flavor model described in this work, where TB mixing is obtained exactly at LO. It is well-known that discrete group Z_N or continuous one like $U(1)$ are usually introduced to eliminate unwanted couplings, to ensure the need vacuum alignment and to reproduce the observed charged lepton mass hierarchies. In the present work, the auxiliary symmetry $Z_4 \times Z_2$ is introduced for this purpose. It is notable that the charged lepton mass hierarchies are determined by the $T_{13} \times Z_4 \times Z_2$ flavor symmetry itself without invoking a Froggatt-Nielsen $U(1)$ symmetry.

This paper is organized as follows. In section 2, we discuss the relevant features of T_{13} group. In section 3, the structure of the model is described, the LO results for neutrino as well

²The vacuum alignment and the next leading order correction are not discussed in Ref. [25], a set of numerical values are chosen by hand for the model parameters so that the resulting lepton masses and flavor mixing are consistent with experimental data.

as charged lepton mass matrices are presented. In section 4, we show how to get in a natural way the required vacuum alignment used throughout the paper. In section 5, we present the study on the corrections introduced by the higher order terms, which is responsible for the deviation from TB mixing. Finally section 6 is devoted to our conclusion. We give the explicit representation matrices and the Clebsch-Gordan coefficients of T_{13} group in Appendix A. The analysis of the subleading corrections to the vacuum alignment is presented in Appendix B.

2 The discrete group T_{13}

The discrete group T_{13} is a subgroup of $SU(3)$, and it is smallest discrete group with two complex irreducible three-dimensional representations. T_{13} is isomorphic to $Z_{13} \rtimes Z_3$ [26,27], consequently it has 39 group elements. T_{13} can be generated by two elements S and T obeying the relations

$$S^{13} = T^3 = 1, \quad ST = TS^3 \quad (2)$$

The 39 elements of the group belong to 7 conjugate classes and are generated from S and T as follows,

$$\begin{aligned} \mathcal{C}_1 &: e \\ \mathcal{C}_2 &: T, TS, TS^2, TS^3, TS^4, TS^5, TS^6, TS^7, TS^8, TS^9, TS^{10}, TS^{11}, TS^{12} \\ \mathcal{C}_3 &: T^2, T^2S, T^2S^2, T^2S^3, T^2S^4, T^2S^5, T^2S^6, T^2S^7, T^2S^8, T^2S^9, T^2S^{10}, T^2S^{11}, T^2S^{12} \\ \mathcal{C}_4 &: S, S^3, S^9 \\ \mathcal{C}_5 &: S^4, S^{10}, S^{12} \\ \mathcal{C}_6 &: S^2, S^5, S^6 \\ \mathcal{C}_7 &: S^7, S^8, S^{11} \end{aligned} \quad (3)$$

The T_{13} group has 7 inequivalent irreducible representations $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$, $\mathbf{3}_1$, $\bar{\mathbf{3}}_1$, $\mathbf{3}_2$ and $\bar{\mathbf{3}}_2$. It is easy to see that the one-dimensional representations are given by

$$\begin{aligned} \mathbf{1}_1 &: S = 1, \quad T = 1 \\ \mathbf{1}_2 &: S = 1, \quad T = \omega \\ \mathbf{1}_3 &: S = 1, \quad T = \omega^2 \end{aligned} \quad (4)$$

where $\omega = e^{2i\pi/3}$. The three-dimensional representations are given by

$$\begin{aligned} \mathbf{3}_1 &: S = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \mathbf{3}_2 &: S = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \quad (5)$$

where $\rho = e^{2i\pi/13}$, the $\bar{\mathbf{3}}_1$ and $\bar{\mathbf{3}}_2$ representations can be obtained by performing the complex conjugation of $\mathbf{3}_1$ and $\mathbf{3}_2$ respectively. We can straightforwardly calculate the character

	classes						
	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6	\mathcal{C}_7
$n_{\mathcal{C}_i}$	1	13	13	3	3	3	3
$h_{\mathcal{C}_i}$	1	3	3	13	13	13	13
$\mathbf{1}_1$	1	1	1	1	1	1	1
$\mathbf{1}_2$	1	ω	ω^2	1	1	1	1
$\mathbf{1}_3$	1	ω^2	ω	1	1	1	1
$\mathbf{3}_1$	3	0	0	ξ_1	ξ_1^*	ξ_2	ξ_2^*
$\bar{\mathbf{3}}_1$	3	0	0	ξ_1^*	ξ_1	ξ_2^*	ξ_2
$\mathbf{3}_2$	3	0	0	ξ_2	ξ_2^*	ξ_1^*	ξ_1
$\bar{\mathbf{3}}_2$	3	0	0	ξ_2^*	ξ_2	ξ_1	ξ_1^*

Table 1: Character table of the T_{13} group, where $\xi_1 = \rho + \rho^3 + \rho^9$, $\xi_2 = \rho^2 + \rho^5 + \rho^6$, $\rho = e^{2i\pi/13}$ and $\omega = e^{2i\pi/3}$. $n_{\mathcal{C}_i}$ denotes the number of the elements contained in the class \mathcal{C}_i , and $h_{\mathcal{C}_i}$ is the order of the elements of \mathcal{C}_i .

table of T_{13} , which is shown in Table 1. Then the multiplication rules between various representations follow immediately,

$$\begin{aligned}
\mathbf{1}_1 \otimes R &= R \otimes \mathbf{1}_1 = R, \quad \mathbf{1}_2 \otimes \mathbf{1}_2 = \mathbf{1}_3, \quad \mathbf{1}_2 \otimes \mathbf{1}_3 = \mathbf{1}_1, \quad \mathbf{1}_3 \otimes \mathbf{1}_3 = \mathbf{1}_2, \\
\mathbf{1}_i \otimes \mathbf{3}_1 &= \mathbf{3}_1, \quad \mathbf{1}_i \otimes \bar{\mathbf{3}}_1 = \bar{\mathbf{3}}_1, \quad \mathbf{1}_i \otimes \mathbf{3}_2 = \mathbf{3}_2, \quad \mathbf{1}_i \otimes \bar{\mathbf{3}}_2 = \bar{\mathbf{3}}_2, \\
\mathbf{3}_1 \otimes \mathbf{3}_1 &= \bar{\mathbf{3}}_{1S} \oplus \bar{\mathbf{3}}_{1A} \oplus \mathbf{3}_2, \quad \mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2, \\
\mathbf{3}_1 \otimes \mathbf{3}_2 &= \mathbf{3}_1 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2, \quad \mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 = \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2, \\
\bar{\mathbf{3}}_1 \otimes \bar{\mathbf{3}}_1 &= \mathbf{3}_{1S} \oplus \mathbf{3}_{1A} \oplus \bar{\mathbf{3}}_2, \quad \bar{\mathbf{3}}_1 \otimes \mathbf{3}_2 = \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2, \\
\bar{\mathbf{3}}_1 \otimes \bar{\mathbf{3}}_2 &= \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2, \quad \mathbf{3}_2 \otimes \mathbf{3}_2 = \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_{2S} \oplus \bar{\mathbf{3}}_{2A}, \\
\mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 &= \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3 \oplus \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1, \quad \bar{\mathbf{3}}_2 \otimes \bar{\mathbf{3}}_2 = \mathbf{3}_1 \oplus \mathbf{3}_{2S} \oplus \mathbf{3}_{2A}
\end{aligned} \tag{6}$$

where the indices $i = 2, 3$, R indicates any T_{13} irreducible representation, and the subscript S and A denote symmetric and anti-symmetric products respectively. The explicit representation matrices of the group elements for the three dimensional irreducible representations are listed in Appendix A. From these representation matrices, one can directly calculate the Clebsch-Gordan coefficients for the decomposition of the product representations, which are given in Appendix A as well.

3 The structure of the model

The model is supersymmetric and based on the discrete symmetry $T_{13} \times Z_4 \times Z_2$. Supersymmetry (SUSY) is introduced in order to simplify the discussion of the vacuum alignment. All the fields of the model, together with their transformation properties under the flavor group, are listed in Table 2. We assign the 3 generation of left-handed lepton doublets ℓ to be the $\mathbf{3}_1$ representation, while the right-handed charged lepton e^c , μ^c and τ^c transform as $\mathbf{1}_1$, $\mathbf{1}_2$ and $\mathbf{1}_3$ respectively. It is notable that the three right-handed neutrinos ν_1^c , ν_2^c and ν_3^c are assigned as $\mathbf{1}_1$, $\mathbf{1}_2$ and $\mathbf{1}_3$ as well, they transform in the same way as the right-handed charged lepton fields. This is an interesting feature of the model. We note that in popular

Fields	ℓ	e^c	μ^c	τ^c	ν_1^c	ν_2^c	ν_3^c	$h_{u,d}$	χ	ξ	ϕ	η	χ^0	ρ^0	θ^0	η^0	ξ^0
T_{13}	$\mathbf{3}_1$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	$\mathbf{1}_1$	$\bar{\mathbf{3}}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_2$	$\mathbf{1}_2$	$\mathbf{1}_3$	$\bar{\mathbf{3}}_2$	$\mathbf{1}_1$
Z_4	1	i	-1	-i	1	1	1	1	i	i	1	1	-1	-1	-1	1	-i
Z_2	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	1	1	1	1	-1
$U(1)_R$	1	1	1	1	1	1	1	0	0	0	0	0	2	2	2	2	2

Table 2: The transformation properties of the matter fields, the electroweak Higgs doublets, the flavon fields and the driving fields under the flavor symmetry $T_{13} \times Z_4 \times Z_2$.

A_4 and S_4 models, the right-handed neutrinos are frequently treated to be a triplet [5, 13]. Lepton masses and mixing arise from the spontaneous breaking of the flavor symmetry by means of the flavon fields, they are neutral under the standard model gauge group and are divided into two sets $\Phi_\ell = \{\chi, \xi\}$ and $\Phi_\nu = \{\phi, \eta\}$. We note that all the flavon fields are triplets under T_{13} in this work, Φ_ℓ is responsible for the charged lepton masses and Φ_ν for the neutrino masses at LO. In the following, we shall discuss the LO predictions for fermion masses and flavor mixings. For the time being we assume that the scalar components of the flavon fields acquire vacuum expectation values (VEV) according to the following scheme,

$$\begin{aligned}
\langle \chi \rangle &= v_\chi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, & \langle \xi \rangle &= v_\xi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
\langle \phi \rangle &= v_\phi \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, & \langle \eta \rangle &= \begin{pmatrix} 0 \\ v_\eta \\ 0 \end{pmatrix}
\end{aligned} \tag{7}$$

In section 4 we shall show that the above alignment is indeed naturally realized at LO from the most general superpotential allowed by the symmetry of the model.

3.1 Charged leptons

The charged lepton masses are described by the following superpotential

$$w_\ell = \sum_{i=1}^5 \frac{y_{e_i}}{\Lambda^3} e^c (\ell \mathcal{O}_i)_{\mathbf{1}_1} h_d + \frac{y'_\mu}{\Lambda^2} \mu^c (\ell (\xi \xi)_{\bar{\mathbf{3}}_{1S}})_{\mathbf{1}_3} h_d + \frac{y_\tau}{\Lambda} \tau^c (\ell \chi)_{\mathbf{1}_2} h_d + \dots \tag{8}$$

where

$$\mathcal{O} = \{(\chi(\chi\chi)_{\bar{\mathbf{3}}_2})_{\bar{\mathbf{3}}_1}, ((\chi\chi)_{\mathbf{3}_{1S}}\xi)_{\bar{\mathbf{3}}_{1S}}, ((\chi\chi)_{\mathbf{3}_{1S}}\xi)_{\bar{\mathbf{3}}_{1A}}, ((\chi\chi)_{\bar{\mathbf{3}}_2}\xi)_{\bar{\mathbf{3}}_1}, (\chi(\xi\xi)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_1}\} \tag{9}$$

We note that the subscripts $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$, $\bar{\mathbf{3}}_1$ etc denote the T_{13} contractions. In the above superpotential w_ℓ , for each charged lepton, only the lowest order operators in the expansion in powers of $1/\Lambda$ are displayed explicitly. Dots stand for higher dimensional operators which will be discussed later. It is remarkable that the Z_4 symmetry imposes different powers of χ and ξ for the electron, muon and tauon terms, i.e., only the tau mass is generated at LO, the muon and the electron masses are generated by high order contributions. After electroweak

and flavor symmetry breaking, we have

$$\begin{aligned}
w_\ell &= [y_{e1} \frac{v_\chi^3}{\Lambda^3} + 4y_{e2} \frac{v_\chi^2 v_\xi}{\Lambda^3} + y_{e4} \frac{v_\chi^2 v_\xi}{\Lambda^3} + y_{e5} \frac{v_\chi v_\xi^2}{\Lambda^3}] v_d e^c (e + \mu + \tau) + 2y'_\mu \frac{v_\xi^2}{\Lambda^2} v_d \mu^c (e + \omega^2 \mu + \omega \tau) \\
&\quad + y_\tau \frac{v_\chi}{\Lambda} v_d \tau^c (e + \omega \mu + \omega^2 \tau) \\
&\equiv y_e \frac{v_\chi^3}{\Lambda^3} v_d e^c (e + \mu + \tau) + y_\mu \frac{v_\xi^2}{\Lambda^2} v_d \mu^c (e + \omega^2 \mu + \omega \tau) + y_\tau \frac{v_\chi}{\Lambda} v_d \tau^c (e + \omega \mu + \omega^2 \tau) \quad (10)
\end{aligned}$$

where $v_d = \langle h_d \rangle$, the parameters y_e and y_μ are parameterized as $y_e = y_{e1} + (4y_{e2} + y_{e4})v_\xi/v_\chi + y_{e5}v_\xi^2/v_\chi^2$ and $y_\mu = 2y'_\mu$. As a result, the charged lepton mass matrix has the form

$$\begin{aligned}
m_\ell &= \begin{pmatrix} y_e \frac{v_\chi^3}{\Lambda^3} & y_e \frac{v_\chi^3}{\Lambda^3} & y_e \frac{v_\chi^3}{\Lambda^3} \\ y_\mu \frac{v_\xi^2}{\Lambda^2} & \omega^2 y_\mu \frac{v_\xi^2}{\Lambda^2} & \omega y_\mu \frac{v_\xi^2}{\Lambda^2} \\ y_\tau \frac{v_\chi}{\Lambda} & \omega y_\tau \frac{v_\chi}{\Lambda} & \omega^2 y_\tau \frac{v_\chi}{\Lambda} \end{pmatrix} v_d \\
&= \begin{pmatrix} y_e \frac{v_\chi^3}{\Lambda^3} & 0 & 0 \\ 0 & y_\mu \frac{v_\xi^2}{\Lambda^2} & 0 \\ 0 & 0 & y_\tau \frac{v_\chi}{\Lambda} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} v_d \quad (11)
\end{aligned}$$

Obviously the charged lepton mass matrix is diagonalized by performing the transformation $\ell \rightarrow U_\ell \ell$, where U_ℓ is

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (12)$$

and the charged lepton masses are given by

$$m_e = \sqrt{3} \left| y_e \frac{v_\chi^3}{\Lambda^3} v_d \right|, \quad m_\mu = \sqrt{3} \left| y_\mu \frac{v_\xi^2}{\Lambda^2} v_d \right|, \quad m_\tau = \sqrt{3} \left| y_\tau \frac{v_\chi}{\Lambda} v_d \right| \quad (13)$$

We see that the charged lepton mass hierarchies are generated by the spontaneous breaking of the flavor symmetry. To estimate the order of magnitudes of v_χ and v_ξ , we can use the experimental data on the ratios of charged lepton masses. Assuming that the coefficients y_e , y_μ and y_τ are of $\mathcal{O}(1)$, we have

$$\begin{aligned}
\frac{m_e}{m_\tau} &\sim \frac{v_\chi^2}{\Lambda^2} \simeq 0.0003 \\
\frac{m_\mu}{m_\tau} &\sim \frac{v_\xi^2}{v_\chi \Lambda} \simeq 0.06 \quad (14)
\end{aligned}$$

These relations are satisfied for

$$\left(\frac{v_\chi}{\Lambda}, \frac{v_\xi}{\Lambda} \right) \sim (0.017, \pm 0.032) \quad (15)$$

we see that the amplitudes of both v_χ/Λ and v_ξ/Λ are roughly of the same order about λ_c^2 , where λ_c is the Cabibbo angle. It is interesting to investigate the flavor symmetry breaking pattern in the charged lepton sector, it is induced by the VEVs of χ and ξ at LO. Given the explicit representation matrices listed in Appendix A, it is obvious that the VEVs of χ and ξ are invariant under the action of T and T^2 . Furthermore, we can check that the hermitian matrix $m_\ell^\dagger m_\ell$ is invariant under both T and T^2 as well. Therefore we conclude that the T_{13} flavor symmetry is broken down to the Z_3 subgroup generated by the element T in the charged lepton sector at LO.

3.2 Neutrinos

The superpotential for the neutrino sector can be written as

$$\begin{aligned}
w_\nu^{SS} &= \frac{y_{\nu 1}}{\Lambda} \nu_1^c(\ell\phi)_{\mathbf{1}_1} h_u + \frac{y_{\nu 2}}{\Lambda} \nu_2^c(\ell\phi)_{\mathbf{1}_3} h_u + \frac{y_{\nu 3}}{\Lambda} \nu_3^c(\ell\phi)_{\mathbf{1}_2} h_u + \frac{1}{2} M_1 \nu_1^c \nu_1^c + \frac{1}{2} M_2 (\nu_2^c \nu_3^c + \nu_3^c \nu_2^c) + \dots \\
w_\nu^{eff} &= \frac{x_{\nu 1}}{\Lambda^3} ((\ell h_u \ell h_u)_{\mathbf{\bar{3}}_{1S}} (\phi\phi)_{\mathbf{3}_{1S}})_{\mathbf{1}_1} + \frac{x_{\nu 2}}{\Lambda^3} ((\ell h_u \ell h_u)_{\mathbf{\bar{3}}_{1S}} (\eta\eta)_{\mathbf{3}_1})_{\mathbf{1}_1} + \frac{x_{\nu 3}}{\Lambda^3} ((\ell h_u \ell h_u)_{\mathbf{3}_2} (\phi\phi)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_1} \\
&\quad + \frac{x_{\nu 4}}{\Lambda^3} ((\ell h_u \ell h_u)_{\mathbf{3}_2} (\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_1} + \dots
\end{aligned} \tag{16}$$

where M_1 and M_2 are constants with dimension of mass, they are naturally of the same order as the cutoff scale Λ , and the factor of $\frac{1}{2}$ is a normalization factor for convenience. We note that w_ν^{SS} denotes the lagrangian of the type I see-saw mechanism, and w_ν^{eff} is the collection of higher dimensional Weinberg operators. Taking into account the vacuum alignment shown in Eq.(7), we can read the Dirac and Majorana mass matrices immediately from w_ν^{SS} as follows

$$m_D = \begin{pmatrix} 0 & y_{\nu 1} & -y_{\nu 1} \\ 0 & \omega^2 y_{\nu 2} & -\omega y_{\nu 2} \\ 0 & \omega y_{\nu 3} & -\omega^2 y_{\nu 3} \end{pmatrix} \frac{v_\phi}{\Lambda} v_u, \quad m_M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_2 \\ 0 & M_2 & 0 \end{pmatrix} \tag{17}$$

where v_u is the vacuum expectation value of the Higgs field h_u . It is remarkable that the eigenvalues of the Majorana mass matrix m_M are M_1 , M_2 and $-M_2$, two of the right handed neutrinos are degenerate at LO. This is a distinguished feature of our model from the previous flavor models in which the right-handed neutrinos are usually treated to form a triplet. It is very interesting to discuss the assignment of right-handed neutrinos as singlets and the corresponding phenomenological implications in flavor models based on A_4 , $\Delta(27)$, S_4 and so on. The light neutrino mass matrix from see-saw mechanism is given by the well-known see-saw formula

$$m_\nu^{SS} = -m_D^T m_M^{-1} m_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a - 2b & a - b \\ 0 & a - b & -a - 2b \end{pmatrix} \frac{v_u^2}{\Lambda} \tag{18}$$

where

$$a = y_{\nu 1}^2 v_\phi^2 / (\Lambda M_1), \quad b = y_{\nu 2} y_{\nu 3} v_\phi^2 / (\Lambda M_2) \tag{19}$$

The superpotential w_ν^{eff} leads to the following effective light neutrino mass matrix

$$m_\nu^{eff} = \begin{pmatrix} r & 0 & 0 \\ 0 & s & t \\ 0 & t & s \end{pmatrix} \frac{v_u^2}{\Lambda} \tag{20}$$

where

$$\begin{aligned}
r &= -2x_{\nu 4} v_\eta v_\phi / \Lambda^2 \\
s &= 2x_{\nu 3} v_\phi^2 / \Lambda^2 \\
t &= -4x_{\nu 1} v_\phi^2 / \Lambda^2 + 2x_{\nu 2} v_\eta^2 / \Lambda^2
\end{aligned} \tag{21}$$

Therefore in the flavor basis where the charged lepton mass matrix is diagonal, the light neutrino mass matrices read

$$\begin{aligned}
m_\nu^{SS} &= \begin{pmatrix} -2b & b & b \\ b & a & -a-b \\ b & -a-b & a \end{pmatrix} \frac{v_u^2}{\Lambda} \\
m_\nu^{eff} &= \begin{pmatrix} r+2s+2t & r-s-t & r-s-t \\ r-s-t & r-s+2t & r+2s-t \\ r-s-t & r+2s-t & r-s+2t \end{pmatrix} \frac{v_u^2}{3\Lambda}
\end{aligned} \tag{22}$$

Both the light neutrino mass matrices m_ν^{SS} and m_ν^{eff} are $2 \leftrightarrow 3$ invariant, and they satisfy the magic symmetry $(m_\nu^{SS(eff)})_{11} + (m_\nu^{SS(eff)})_{13} = (m_\nu^{SS(eff)})_{22} + (m_\nu^{SS(eff)})_{32}$. Therefore they are exactly diagonalized by the TB mixing matrix

$$\begin{aligned}
U_{TB} m_\nu^{SS} U_{TB} &= \text{diag}(-3b, 0, 2a+b) \frac{v_u^2}{\Lambda} \\
U_{TB} m_\nu^{eff} U_{TB} &= \text{diag}(s+t, r, -s+t) \frac{v_u^2}{\Lambda}
\end{aligned} \tag{23}$$

We note that the contribution m_ν^{SS} from the see-saw mechanism is of the same order as m_ν^{eff} coming from the Weinberg operators, consequently both contributions should be included. The light neutrino mass matrix is the sum of m_ν^{SS} and m_ν^{eff}

$$m_\nu = m_\nu^{SS} + m_\nu^{eff} \tag{24}$$

Obviously m_ν is still diagonalized by the TB mixing matrix, and the light neutrino masses are given by

$$\begin{aligned}
m_{\nu 1} &= (s+t-3b) \frac{v_u^2}{\Lambda} \\
m_{\nu 2} &= r \frac{v_u^2}{\Lambda} \\
m_{\nu 3} &= (-s+t+2a+b) \frac{v_u^2}{\Lambda}
\end{aligned} \tag{25}$$

where U_{TB} is the well-known TB mixing matrix

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{26}$$

We note that the contributions proportional to a and b can be absorbed into s and t by redefinition $s \rightarrow s - a - 2b$ and $t \rightarrow t + a - b$, therefore the light neutrino masses depend on three unrelated complex parameters. There are more freedoms to tune the mass differences and then satisfy the constraints associated to neutrino oscillation, the neutrino mass spectrum can be normal hierarchy or inverted hierarchy. In contrast with some "constrained" flavor models, no neutrino mass sum-rules [28] can be found in this model. We could certainly remove the right-handed neutrinos from our model, then the neutrino masses are described by the Weinberg operators w_ν^{eff} , the above conclusions remain invariant. However, if we only

concentrate on the see-saw realization w_ν^{SS} , the second neutrino would be massless although the lepton mixing is of TB form, this scenario is ruled out by the experimental observations.

It is notable that the VEVs of ϕ and η are always changed under the action of any T_{13} group element except unit element, consequently the flavor symmetry T_{13} is broken down to nothing in the neutrino sector at LO. Reminding that ones usually break the flavor symmetry into the low energy neutrino symmetry group Klein four [29–31] or Z_2 [32, 33] to guarantee TB mixing for neutrinos, it is really amazing we can still obtain TB mixing even if the flavor symmetry is broken completely in the neutrino sector at LO.

In short summary, at the LO the T_{13} flavor symmetry is broken down to Z_3 subgroup and nothing in the charged lepton and neutrino sectors respectively. This breaking chain lets us to find the TB scheme at LO as the lepton mixing matrix. However, the mixing angles generally deviate from the TB values after the corrections of the higher order terms are included. It is remarkable that this symmetry breaking pattern of our model has not been studied, as far as we know. It is attractive to investigate whether we can still reproduce TB mixing in models with A_4 or S_4 symmetry, if the flavor symmetry is broken completely in the neutrino sector at LO.

4 Vacuum alignment

The vacuum alignment problem of the model can be solved by the supersymmetric driving fields method introduced in Ref. [33]. This approach exploits a continuous $U(1)_R$ symmetry under which matter fields have $R = +1$, while Higgses and flavon fields have $R = 0$. Such a symmetry will be eventually broken down to the R-parity by small SUSY breaking effects which can be neglected in the first approximation in our analysis. The spontaneous breaking of T_{13} can be employed by introducing the so-called driving fields with $R = 2$, which enter linearly into the superpotential. Five driving fields χ^0 , ρ^0 , θ^0 , η^0 and ξ^0 are introduced in our model, their transformation rules under the flavor symmetry are shown in Table 2. We note that the driving fields ρ^0 and θ^0 are necessary to stabilize the vacuum alignment under subleading corrections. At LO, the most general superpotential dependent on the driving fields, which is invariant under the flavor symmetry group $T_{13} \times Z_4 \times Z_2$, is given by

$$w_v = f_1(\chi^0(\chi\chi)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_1} + f_2(\chi^0(\chi\xi)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_1} + f_3\rho^0(\chi\xi)_{\mathbf{1}_3} + f_4\theta^0(\chi\xi)_{\mathbf{1}_2} + g_1(\eta^0(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_1} + g_2(\eta^0(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_1} + h\xi^0(\xi\phi)_{\mathbf{1}_1} \quad (27)$$

In the SUSY limit, the vacuum configuration is determined by the vanishing of the derivative of w_v with respect to each component of the driving fields

$$\frac{\partial w_v}{\partial \chi_1^0} = f_1\chi_1^2 + f_2\chi_2\xi_1 = 0 \quad (28a)$$

$$\frac{\partial w_v}{\partial \chi_2^0} = f_1\chi_2^2 + f_2\chi_3\xi_2 = 0 \quad (28b)$$

$$\frac{\partial w_v}{\partial \chi_3^0} = f_1\chi_3^2 + f_2\chi_1\xi_3 = 0 \quad (28c)$$

$$\frac{\partial w_v}{\partial \rho^0} = f_3(\chi_1\xi_1 + \omega^2\chi_2\xi_2 + \omega\chi_3\xi_3) = 0 \quad (29)$$

$$\frac{\partial w_v}{\partial \theta^0} = f_4(\chi_1\xi_1 + \omega\chi_2\xi_2 + \omega^2\chi_3\xi_3) = 0 \quad (30)$$

The above equations are satisfied by the alignment

$$\langle \chi \rangle = v_\chi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \xi \rangle = v_\xi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (31)$$

with the relation

$$v_\chi = -\frac{f_2}{f_1}v_\xi, \quad v_\xi \text{ undetermined} \quad (32)$$

Without assuming any fine-tuning among the parameters f_1 and f_2 , the VEVs v_χ and v_ξ are expected to be of the same order of magnitude, this is consistent with the conclusion drew from the charged lepton mass hierarchies. We note that if one component of χ or ξ has vanishing VEV, e.g. $\langle \xi_1 \rangle = 0$, Eqs.(28a)-(28c) imply $\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle = 0$. This means that the VEV of any component of the flavons χ or ξ should be non-zero in order to obtain a non-trivial vacuum configuration. As has been shown in the previous section, at LO the T_{13} flavor symmetry is spontaneously broken by the VEVs of ϕ and η in the neutrino sector, their vacuum configurations are determined by

$$\frac{\partial w_v}{\partial \eta_1^0} = 2g_1\eta_2\eta_3 + g_2\phi_3\eta_1 = 0 \quad (33a)$$

$$\frac{\partial w_v}{\partial \eta_2^0} = 2g_1\eta_1\eta_3 + g_2\phi_1\eta_2 = 0 \quad (33b)$$

$$\frac{\partial w_v}{\partial \eta_3^0} = 2g_1\eta_1\eta_2 + g_2\phi_2\eta_3 = 0 \quad (33c)$$

$$\frac{\partial w_v}{\partial \xi^0} = h(\xi_1\phi_1 + \xi_2\phi_2 + \xi_3\phi_3) = 0 \quad (34)$$

The first three equations Eq.(33a)-(33c) lead to two un-equivalent vacuum configurations ³, the first is

$$\langle \phi \rangle = v_\phi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta \rangle = v_\eta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (37)$$

with

$$v_\eta = -\frac{g_2}{2g_1}v_\phi, \quad v_\phi \text{ undetermined} \quad (38)$$

The second solution is

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v_{\phi_2} \\ v_{\phi_3} \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ v_\eta \\ 0 \end{pmatrix} \quad (39)$$

³We note that the equations can be satisfied by two additional solutions as well. One is

$$\langle \phi \rangle = \begin{pmatrix} v_{\phi_1} \\ 0 \\ v_{\phi_3} \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ 0 \\ v_\eta \end{pmatrix} \quad (35)$$

Another one is

$$\langle \phi \rangle = \begin{pmatrix} v_{\phi_1} \\ v_{\phi_2} \\ 0 \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} v_\eta \\ 0 \\ 0 \end{pmatrix} \quad (36)$$

where v_η , v_{ϕ_1} , v_{ϕ_2} and v_{ϕ_3} are undetermined. However, the above two solutions can be obtained by acting on the vacuum Eq.(39) with the elements T and T^2 respectively. Therefore these two solutions are equivalent to the configuration in Eq.(39).

where v_η , v_{ϕ_2} and v_{ϕ_3} are constrained. Using the alignment of χ in Eq.(31), for the first solution shown in Eq.(37), we can immediately infer from Eq.(34)

$$v_\xi v_\phi = 0 \quad (40)$$

We are led to the trivial solutions $v_\chi = v_\xi = 0$ or $v_\phi = v_\eta = 0$, which can be removed by the interplay of radiative corrections to the scalar potential and soft SUSY breaking terms for the flavon fields. Therefore we choose the second solution in this work, this vacuum configuration can produce the results in the previous section. Then the minimization equation Eq.(34) implies

$$v_{\phi_2} + v_{\phi_3} = 0 \quad (41)$$

This indicates that $\langle\phi_2\rangle$ and $\langle\phi_3\rangle$ have to be equal up to a relative sign, thus ϕ is fully aligned as

$$\langle\phi\rangle = v_\phi \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (42)$$

Starting from the vacuum configurations given in Eq.(31), Eq.(39) and Eq.(42) and acting on them with the elements of the flavor symmetry group T_{13} , we can generate other minima of the scalar potential. However, these new minima are physically equivalent to the original one, it is not restrictive to analyze the model by choosing the vacuum in Eqs.(31,39,42) as local minimum. It is important to check the stability of the LO vacuum configuration, if we introduce small perturbations to the VEVs of the flavon fields as follows,

$$\begin{aligned} \langle\chi\rangle &= v_\chi \begin{pmatrix} 1+x_1 \\ 1+x_2 \\ 1+x_3 \end{pmatrix}, & \langle\xi\rangle &= v_\xi \begin{pmatrix} 1 \\ 1+y_2 \\ 1+y_3 \end{pmatrix} \\ \langle\phi\rangle &= v_\phi \begin{pmatrix} z_1 \\ 1 \\ -1+z_3 \end{pmatrix}, & \langle\eta\rangle &= v_\eta \begin{pmatrix} w_1 \\ 1 \\ w_3 \end{pmatrix} \end{aligned} \quad (43)$$

After some straightforward algebra, we find that the only solution to the minimization equations is

$$\begin{aligned} x_1 = x_2 = x_3 = 0, & \quad y_2 = y_3 = 0 \\ z_1 = z_3 = 0, & \quad w_1 = w_3 = 0 \end{aligned} \quad (44)$$

Therefore the LO vacuum alignment is stable, then we turn to consider the magnitudes of flavon VEVs. Since the VEVs of ϕ and η are closely related with each other through the equations Eqs.(33a)-(33c), and they have the same charges under the auxiliary symmetry $Z_4 \times Z_2$, we expect a common order of magnitude for the VEVs v_χ and v_η . However, the VEVs of $\Phi_\ell = \{\chi, \xi\}$ and $\Phi_\nu = \{\phi, \eta\}$ can be in principle different and they are subject to phenomenological constraints. As we have shown in section 3.1, $\langle\Phi_\ell\rangle$ is responsible for the charged lepton mass hierarchies, and it is required to satisfy

$$\varepsilon \equiv \frac{v_\chi}{\Lambda} \sim \frac{v_\xi}{\Lambda} \sim \lambda_c^2 \quad (45)$$

Among the three neutrino mixing angles, the solar neutrino mixing angle θ_{12} is measured most precisely so far, the experimentally allowed departures of θ_{12} from its TB value $\sin^2 \theta_{12} =$

$1/3$ are at most of order λ_c^2 [1–3]. It is well-known that the superpotentials w_ℓ , w_ν^{SS} , w_ν^{eff} and w_v are corrected by higher dimensional operators in the expansion (please see section 5 and Appendix B for detail), which mostly can be constructed by including the combination $\Phi_\nu \Phi_\nu$ on top of each LO term, thus all the three mixing angles receive corrections of order $\langle \Phi_\nu \rangle^2 / \Lambda^2$ (please see section 5 for detail). Requiring that the mixing angles particular θ_{12} lie in the ranges allowed by neutrino oscillation data, we obtain the condition

$$\varepsilon' \equiv \frac{v_\phi}{\Lambda} \sim \frac{v_\eta}{\Lambda} \leq \lambda_c \quad (46)$$

The same condition follows from the requirement that the generated charged lepton mass hierarchies should be stable under subleading corrections. As a result, we can tolerate a moderate hierarchy between ε and ε' because of the strong constraint of the auxiliary symmetry $Z_4 \times Z_2$. It is a general conclusion that a hierarchy between the VEVs of the flavon fields can be accommodated in a "fully" separated scalar potential. This type of vacuum alignment is usually constructed to generate a large reactor angle [34, 35], i.e. $\theta_{13} \sim \lambda_c$, although it is predicted to be exactly zero at LO. However, the subleading corrections to θ_{13} turn out to be of order λ_c^2 in our model, as we shall demonstrate in next section.

5 Subleading corrections

It is crucial to guarantee that the successful LO predictions are not spoiled by subleading corrections, we will discuss this important issue in detail. The superpotentials w_ℓ , w_ν^{SS} , w_ν^{eff} and w_v are corrected by higher dimensional operators, which arise from adding the products $\Phi_\nu \Phi_\nu$, invariant combination under $Z_4 \times Z_2$, on top of the LO terms. Then the residual Z_3 symmetry in the charged lepton sector would be broken completely by the subleading contributions. The lepton masses and mixing matrices are corrected by both the shift of the vacuum configuration and the higher dimensional operators in the Yukawa superpotentials. As a result, the mass matrices with subleading corrections can be obtained by inserting the modified vacuum alignment into the LO Yukawa operators plus the contributions of the new higher dimensional operators evaluated with the unperturbed VEVs.

The subleading corrections to the vacuum alignment are discussed in Appendix B in detail. The inclusion of the higher dimensional operators in the driving superpotential w_v results in a shift of the VEVs of the flavon fields, the vacuum configuration is modified into

$$\begin{aligned} \langle \chi \rangle &= \begin{pmatrix} v_\chi + \delta v_{\chi 1} \\ v_\chi + \delta v_{\chi 2} \\ v_\chi + \delta v_{\chi 3} \end{pmatrix}, & \langle \xi \rangle &= \begin{pmatrix} v_\xi + \delta v_{\xi 1} \\ v_\xi + \delta v_{\xi 2} \\ v_\xi \end{pmatrix}, \\ \langle \phi \rangle &= \begin{pmatrix} \delta v_{\phi 1} \\ v_\phi + \delta v_{\phi 2} \\ -v_\phi \end{pmatrix}, & \langle \eta \rangle &= \begin{pmatrix} \delta v_{\eta 1} \\ v_\eta \\ \delta v_{\eta 3} \end{pmatrix} \end{aligned} \quad (47)$$

where v_ξ , v_ϕ and v_η remain undetermined, and all the shifts are of order ε'^2 with respect to the LO VEVs, as is shown in Appendix B. Moreover, all components of $\langle \chi \rangle$, $\langle \xi \rangle$, $\langle \phi \rangle$ and $\langle \eta \rangle$ receive different corrections so that the LO alignment is tilted.

5.1 Corrections to the charged lepton mass matrix

In the charged lepton sector, w_ℓ is corrected by the following operators

$$e^c \ell \Phi_\ell^3 \Phi_\nu^2 h_d / \Lambda^5, \quad \mu^c \ell \Phi_\ell^2 \Phi_\nu^2 h_d / \Lambda^4, \quad \tau^c \ell \Phi_\ell \Phi_\nu^2 h_d / \Lambda^3 \quad (48)$$

where all possible contractions among fields are understood. After lengthy and tedious calculations, we find that each element of charged lepton mass matrix gets a small correction. Concretely the corrections to the e row, μ row and τ row are of order $\varepsilon^3 \varepsilon'^2 v_d$, $\varepsilon^2 \varepsilon'^2 v_d$ and $\varepsilon \varepsilon'^2 v_d$ respectively. As a result, the charged lepton mass matrix with subleading corrections can be parameterized as

$$m_\ell = \begin{pmatrix} y_e \varepsilon^2 & y_e \varepsilon^2 & y_e \varepsilon^2 \\ y_\mu \varepsilon & \omega^2 y_\mu \varepsilon & \omega y_\mu \varepsilon \\ y_\tau & \omega y_\tau & \omega^2 y_\tau \end{pmatrix} \varepsilon v_d + \begin{pmatrix} a_{11}^\ell \varepsilon^2 & a_{12}^\ell \varepsilon^2 & a_{13}^\ell \varepsilon^2 \\ a_{21}^\ell \varepsilon & \omega^2 a_{22}^\ell \varepsilon & \omega a_{23}^\ell \varepsilon \\ a_{31}^\ell & \omega a_{32}^\ell & \omega^2 a_{33}^\ell \end{pmatrix} \varepsilon \varepsilon'^2 v_d \quad (49)$$

where the first term denotes the LO contributions, and the second term represents the corrections induced by the higher dimensional operators in Eq.(48). The coefficients a_{ij}^ℓ ($i, j = 1, 2, 3$) are complex numbers with absolute value of order one, their specific values are not determined by the flavor symmetry. Furthermore, we have to consider the corrections from the shifted vacuum alignment. Since the shifts δv_{χ_i} and δv_{ξ_i} are of order $\varepsilon'^2 v_\chi$ and $\varepsilon'^2 v_\xi$ respectively, and the corrections to each matrix element contain one additional power of $\delta v_{\chi_i}/v_\chi$ or $\delta v_{\xi_i}/v_\xi$. Consequently, including these corrections only amounts to a redefinition of the a_{ij}^ℓ parameter in Eq.(49). As a result, the unitary matrix U_ℓ ⁴, which corresponds to the transformation of the charged leptons used to diagonalized m_ℓ , is modified into

$$U_\ell = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} U'_\ell \quad (50)$$

where U'_ℓ is given by

$$U'_\ell = \begin{pmatrix} 1 & (A_\ell \varepsilon'^2)^* & (B_\ell \varepsilon'^2)^* \\ -A_\ell \varepsilon'^2 & 1 & (C_\ell \varepsilon'^2)^* \\ -B_\ell \varepsilon'^2 & -C_\ell \varepsilon'^2 & 1 \end{pmatrix} \quad (51)$$

with

$$\begin{aligned} A_\ell &= (a_{21}^\ell + \omega^2 a_{22}^\ell + \omega a_{23}^\ell) / (3y_\mu) \\ B_\ell &= (a_{31}^\ell + \omega a_{32}^\ell + \omega^2 a_{33}^\ell) / (3y_\tau) \\ C_\ell &= (a_{31}^\ell + \omega^2 a_{32}^\ell + \omega a_{33}^\ell) / (3y_\tau) \end{aligned} \quad (52)$$

The charged lepton masses are corrected by terms of relative order ε'^2 with respect to LO result, therefore the charged lepton mass hierarchies predicted at LO are not spoiled by subleading corrections.

⁴ U_ℓ is the unitary matrix which diagonalizes the hermitian matrix $m_\ell^\dagger m_\ell$.

5.2 Corrections to the neutrino mass matrix

First of all we focus on the corrections to the right-handed Majorana neutrino mass. We note that the modified vacuum alignment doesn't affect the Majorana mass at all, since flavon fields are not involved in the LO Majorana mass terms. The subleading corrections from higher dimensional operators are of the form $\nu_i^c \nu_j^c \Phi_\nu^4 / \Lambda^3$, thus every entry of right-handed Majorana neutrino mass matrix receives corrections of order $\varepsilon'^4 \Lambda$ instead of $\varepsilon'^2 \Lambda$, which can be safely neglected. Then we move to consider the corrections to the neutrino Dirac mass. Among the independent terms of the type $\nu_i^c \ell \Phi_\nu^3 h_u / \Lambda^3$, only the operators $(\nu_i^c \ell \phi^3 h_u)_{11} / \Lambda^3$ give non-zero contributions. As a consequence, the first and the third columns of Dirac mass matrix receive corrections of order $\varepsilon'^3 v_u$. In addition to this correction, inserting the VEV shifts in the LO operators introduces independent corrections of order $\varepsilon'^3 v_u$ to the first and second column elements of the Dirac mass matrix. Including the above two kinds of corrections mentioned, we conclude that all the elements of Dirac mass matrix are corrected by terms of $\mathcal{O}(\varepsilon'^3 v_u)$. With these results, we find that each entry of m_ν^{SS} except the (11) element receives corrections of order $\varepsilon'^4 v_u^2 / \Lambda$. Now we discuss the corrections to the Weinberg operators. The superpotential w_ν^{eff} is corrected by the contraction

$$(\ell h_u \ell h_u \Phi_\nu^4)_{11} / \Lambda^5 \quad (53)$$

Taking into account the contributions of the modified vacuum alignment in addition, we find all the elements of m_ν^{eff} receive corrections of order $\varepsilon'^4 v_u / \Lambda$. As a result, the overall correction to the light neutrino mass matrix is a most general symmetric matrix of order $\varepsilon'^4 v_u / \Lambda$. The neutrino mass matrix including subleading corrections can be written as

$$m_\nu = \varepsilon'^2 \begin{pmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & z & y \end{pmatrix} \frac{v_u^2}{\Lambda} + \varepsilon'^4 \begin{pmatrix} a_{11}^\nu & a_{12}^\nu & a_{13}^\nu \\ a_{12}^\nu & a_{22}^\nu & a_{23}^\nu \\ a_{13}^\nu & a_{23}^\nu & a_{33}^\nu \end{pmatrix} \frac{v_u^2}{\Lambda} \quad (54)$$

where the parameters x, y and z can be easily reconstructed from the LO couplings in Eq.(16), and the coefficients a_{ij}^ν are $\mathcal{O}(1)$ unspecified constants. The matrix m_ν is diagonalized by the unitary transformation

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \end{pmatrix} U'_\nu \quad (55)$$

where U'_ν is close to an identity matrix with small corrections on off-diagonal elements, it is given by

$$U'_\nu = \begin{pmatrix} 1 & A_\nu \varepsilon'^2 & B_\nu \varepsilon'^2 \\ -(A_\nu \varepsilon'^2)^* & 1 & C_\nu \varepsilon'^2 \\ -(B_\nu \varepsilon'^2)^* & -(C_\nu \varepsilon'^2)^* & 1 \end{pmatrix} \quad (56)$$

with

$$\begin{aligned} A_\nu &= \frac{(y^* + z^*)(a_{12}^\nu + a_{13}^\nu) + x(a_{12}^{\nu*} + a_{13}^{\nu*})}{\sqrt{2}[|x|^2 - |y + z|^2]} \\ B_\nu &= \frac{i(y^* + z^*)(a_{22}^\nu - a_{33}^\nu) + i(y - z)(a_{22}^{\nu*} - a_{33}^{\nu*})}{-4(yz^* + y^*z)} \\ C_\nu &= \frac{ix^*(a_{12}^\nu - a_{13}^\nu) + i(y - z)(a_{12}^{\nu*} - a_{13}^{\nu*})}{\sqrt{2}[|y - z|^2 - |x|^2]} \end{aligned} \quad (57)$$

The PMNS matrix is $U_{PMNS} = U_\ell^\dagger U_\nu$, then the parameters of the lepton mixing matrix are modified as

$$\begin{aligned}\sin \theta_{13} &= \left| \frac{1}{\sqrt{3}}(\sqrt{2}B_\nu + C_\nu)\varepsilon'^2 - \frac{1}{\sqrt{2}}(A_\ell^* - B_\ell^*)(\varepsilon'^2)^* \right| \\ \sin^2 \theta_{12} &= \frac{1}{3} + \left[\frac{1}{3}(\sqrt{2}A_\nu - A_\ell - B_\ell)\varepsilon'^2 + c.c. \right] \\ \sin^2 \theta_{23} &= \frac{1}{2} + \left[\left(-\frac{1}{2\sqrt{3}}B_\nu + \frac{1}{\sqrt{6}}C_\nu + \frac{1}{2}C_\ell\right)\varepsilon'^2 + c.c. \right]\end{aligned}\quad (58)$$

We see that all the three mixing angles receive corrections of order ε'^2 from both the neutrino and the charged lepton sectors. As is pointed out in section 4, the data on solar neutrino mixing angle θ_{12} constrain $\varepsilon' \leq \lambda_c$. Then the reactor angle θ_{13} is of order λ_c^2 , it is within the sensitivity of the experiments which are now in preparation and will take data in the near future [36, 37]. Since three complex parameters which are related with three light neutrino masses are involved at LO, the light neutrino mass spectrum can be normal hierarchy or inverted hierarchy, and the phenomenological predictions of the model are just the generic results of neutrino mass matrix with TB mixing, e.g., degenerate neutrino mass spectrum is disfavored since strong fine-tuning is required to produce the observed mass squared differences Δm_{sol}^2 and Δm_{atm}^2 , and the $0\nu 2\beta$ -decay mass $|m_{ee}|$ in inverted hierarchy is generally larger than that in normal hierarchy.

6 Phenomenological implications

In the following, we shall investigate the physical consequences of our model, and the corresponding predictions are presented. We perform a numerical analysis by treating all the LO and NLO coefficients as random complex numbers with absolute value between 1/3 and 3, the expansion parameter ε and ε' are set to the indicative values 0.05 and 0.22 respectively. In Fig.1, we plot the effective $0\nu 2\beta$ -decay mass $|m_{ee}|$ as a function of the lightest neutrino mass. The constraints which have been imposed to draw the points are the experimental values at 3σ for the neutrino oscillation parameters Δm_{atm}^2 , Δm_{sol}^2 , $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$ and $\sin^2 \theta_{13}$ [1–3]. We also show the future sensitivity on the lightest neutrino mass of 0.2 eV from the KATRIN experiment [38], and the horizontal lines represent the sensitivities of the future $0\nu 2\beta$ -decay experiments CUORE [39] and Majorana [40]/GERDA III [41], which are 15 meV and 20 meV respectively. It is obvious that the effective mass $|m_{ee}|$ of inverted hierarchy (IH) is generally larger than that of normal hierarchy (NH). Since the bulk of data are predicted to be above the sensitivity of CUORE experiment for IH, the rare process $0\nu 2\beta$ -decay should be observable in future, if the neutrino spectrum is IH. We note that most of the points fall into the region where the lightest neutrino mass is smaller than 0.02 eV for NH spectrum, and a large set of points lie in the region of the lightest neutrino mass between 0.01 eV and 0.04 eV for IH case. The values beyond these regions, in particular the region of degenerate spectrum, are strongly disfavored.

Finally, we show the sum of light neutrino mass as a function of the lightest neutrino mass in Fig.2. The vertical line denotes the future sensitivity of KATRIN experiment, and the horizontal lines are the cosmological bounds [42]. The first one is at 0.60 eV, which is obtained by combining the data in Ref. [43], and the second one at 0.19 eV corresponds to all the previous data combined to the small scale primordial spectrum from Lyman-alpha

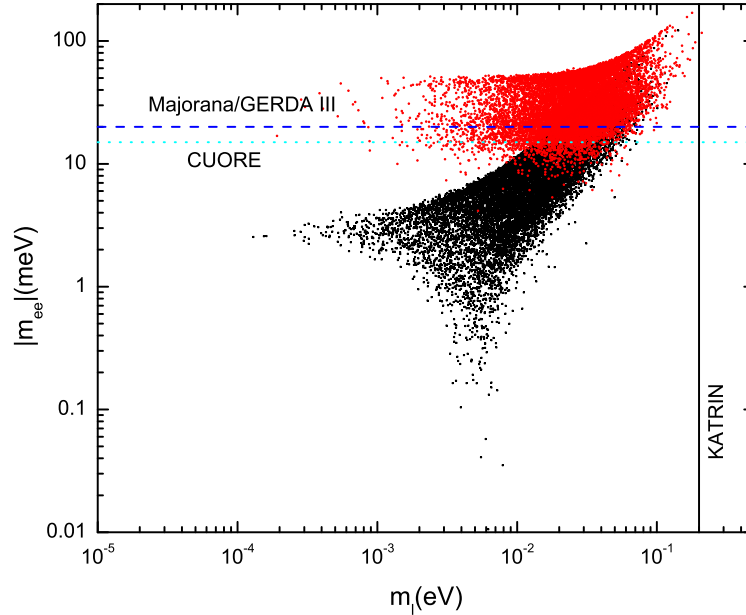


Figure 1: The effective mass $|m_{ee}|$ as a function of the lightest neutrino mass. The red corresponds to the inverted hierarchy neutrino mass spectrum, and the black corresponds to the normal hierarchy case. The future sensitivity of 0.2 eV of KATRIN experiment is shown by the vertical solid line, the future expected bounds on $|m_{ee}|$ from CUORE and Majorana/GERDA III experiments are represented by horizontal lines.

(Ly α) forest clouds [44]. We see that the current cosmological information on the sum of the neutrino masses can hardly distinguish the NH spectrum from the IH spectrum. However, such a discrimination could be possible if these bounds are improved in the near future.

7 Conclusions and discussions

In this work, we have presented a T_{13} model for TB mixing based on the flavor symmetry $T_{13} \times Z_4 \times Z_2$. Both the charged lepton singlets e^c , μ^c , τ^c and the right-handed neutrinos ν_i^c are assigned as T_{13} singlets in this work. The light neutrino masses are generated as a combination of type I see-saw mechanism and Weinberg operators, and neutrino mass spectrum can be normal hierarchy or inverted hierarchy. In the charged lepton sector, the flavon fields $\Phi_\ell = \{\chi, \xi\}$ break the T_{13} group into the Z_3 subgroup at LO, and the symmetry breaking parameter $\varepsilon \equiv \langle \chi \rangle / \Lambda \sim \langle \xi \rangle / \Lambda$ controls the charged lepton mass hierarchies without invoking a Froggatt-Nielsen $U(1)$ symmetry. In the neutrino sector, the T_{13} group is entirely broken by the flavon fields $\Phi_\nu = \{\phi, \eta\}$ at LO, the symmetry breaking parameter $\varepsilon' \equiv \langle \phi \rangle / \Lambda \sim \langle \eta \rangle / \Lambda$ can be chosen to be of the order of Cabibbo angle λ_c without spoiling the LO predictions and vacuum alignment. It is a noticeable feature that we can still reproduce the TB mixing even if the flavor symmetry is broken to nothing in the neutrino sector at LO.

The subleading corrections are discussed in detail. The subleading operators contributing to lepton mass and vacuum alignment are obtained by inserting Φ_ν^2 into the LO operators

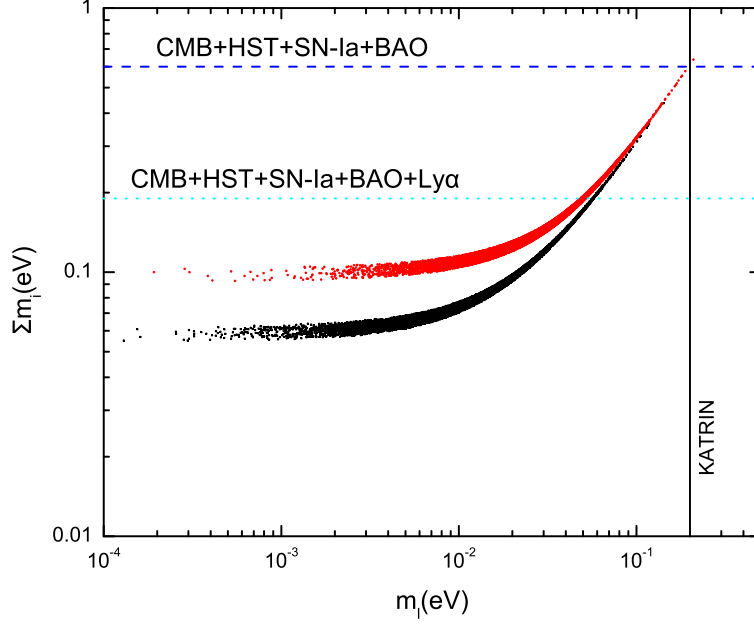


Figure 2: The effective mass $|m_{ee}|$ as a function of the lightest neutrino mass. The red corresponds to the inverted hierarchy neutrino mass spectrum, and the black corresponds to the normal hierarchy case. The vertical solid line represents the future sensitivity of 0.2 eV from the KATRIN experiment, and the horizontal lines refer to the cosmological bounds.

in all possible ways and by extracting the $T_{13} \times Z_4 \times Z_2$ invariants. We have showed that all the mixing angles receive corrections at the level of $\mathcal{O}(\lambda_c^2)$, in particular, the reactor angle θ_{13} is predicted to be within the reach of next generation neutrino oscillation experiments, although it is small. Furthermore, since the neutrino mass matrix depends on three unrelated complex parameters at LO, the phenomenological consequences of the model are the general results of neutrino mass matrix with TB mixing, there are no model-dependent peculiar predictions.

In the end, we discuss whether we can extend the T_{13} flavor symmetry from the neutrino sector to the quark sector. The most naive way is to adopt for quarks the same classification scheme under $T_{13} \times Z_4 \times Z_2$ that we have used for leptons. With such an assignment, both up and down type quark mass matrices are diagonalized by the same unitary matrix U_ℓ shown in Eq.(12), as a consequence the CKM matrix is a unit matrix at LO, this is a good first order approximation. The off-diagonal elements of the CKM matrix arise when the subleading contributions are taken into account. As has been showed in section 5, the subleading corrections to the three quark mixing angles are of order λ_c^2 , the resulting CKM matrix should have the same form of the unitary matrix U'_ℓ given in Eq.(51). Therefore it seems difficult to reproduce the quark mixing without introducing new ingredients in the symmetry breaking sector. Furthermore, there is another lack if we adopt for quark the same structure as that in the lepton sector, the resulting mass hierarchies among the up type quarks are not realistic, although it is a satisfactory result that the mass spectrums of down type quarks and charged leptons are predicted to have the same pattern. Since the top quark mass is of order of the electroweak symmetry breaking scale, it is much heavier

than the remaining quarks, it is natural to assign quarks as 2+1 representation instead of a triplet. In the context of $U(2)$ flavor group, this assignment has been known to give realistic quark mixing matrix and mass hierarchies [45]. Inspired by this assignment, it is usually suggested to extend the flavor symmetry group, which can produce the neutrino TB mixing, to its double covering in order to give a coherent description of all fermion masses and flavor mixings. The flavor models based on T' and \mathcal{I}' groups [12, 46], which are the double covering groups of A_4 and A_5 respectively, have been studied extensively. These models can really lead to a good description of the observed pattern of quark masses and mixing besides reproducing TB mixing (or the gold ratio mixing pattern) for neutrinos. Following the same logic, we expect the double covering group of T_{13} can simultaneously describe the lepton and quark sector very well.

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Appendix A: Representation matrices and Clebsch-Gordan coefficients of T_{13}

The T_{13} group has 7 inequivalent irreducible representations $\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3, \mathbf{3}_1, \bar{\mathbf{3}}_1, \mathbf{3}_2$ and $\bar{\mathbf{3}}_2$. The representation matrices of the generators S and T in these representations are given in Eq.(4) and Eq.(5). In the following, we present the representation matrices of all the group elements for the three dimensional representations. The explicit expressions of the elements in the $\mathbf{3}_1$ representation are

$$\begin{aligned}
 \mathcal{C}_1 : \quad e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 \mathcal{C}_2 : \quad T &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & 0 & \rho^9 \\ \rho & 0 & 0 \\ 0 & \rho^3 & 0 \end{pmatrix}, \quad TS^2 = \begin{pmatrix} 0 & 0 & \rho^5 \\ \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \end{pmatrix}, \\
 TS^3 &= \begin{pmatrix} 0 & 0 & \rho \\ \rho^3 & 0 & 0 \\ 0 & \rho^9 & 0 \end{pmatrix}, \quad TS^4 = \begin{pmatrix} 0 & 0 & \rho^{10} \\ \rho^4 & 0 & 0 \\ 0 & \rho^{12} & 0 \end{pmatrix}, \quad TS^5 = \begin{pmatrix} 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix} \\
 TS^6 &= \begin{pmatrix} 0 & 0 & \rho^2 \\ \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \end{pmatrix}, \quad TS^7 = \begin{pmatrix} 0 & 0 & \rho^{11} \\ \rho^7 & 0 & 0 \\ 0 & \rho^8 & 0 \end{pmatrix}, \quad TS^8 = \begin{pmatrix} 0 & 0 & \rho^7 \\ \rho^8 & 0 & 0 \\ 0 & \rho^{11} & 0 \end{pmatrix} \\
 TS^9 &= \begin{pmatrix} 0 & 0 & \rho^3 \\ \rho^9 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix}, \quad TS^{10} = \begin{pmatrix} 0 & 0 & \rho^{12} \\ \rho^{10} & 0 & 0 \\ 0 & \rho^4 & 0 \end{pmatrix}, \quad TS^{11} = \begin{pmatrix} 0 & 0 & \rho^8 \\ \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \end{pmatrix}, \\
 TS^{12} &= \begin{pmatrix} 0 & 0 & \rho^4 \\ \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \end{pmatrix} \\
 \mathcal{C}_3 : \quad T^2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^2S = \begin{pmatrix} 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \\ \rho & 0 & 0 \end{pmatrix}, \quad T^2S^2 = \begin{pmatrix} 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \\ \rho^2 & 0 & 0 \end{pmatrix}, \\
 T^2S^3 &= \begin{pmatrix} 0 & \rho^9 & 0 \\ 0 & 0 & \rho \\ \rho^3 & 0 & 0 \end{pmatrix}, \quad T^2S^4 = \begin{pmatrix} 0 & \rho^{12} & 0 \\ 0 & 0 & \rho^{10} \\ \rho^4 & 0 & 0 \end{pmatrix}, \quad T^2S^5 = \begin{pmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \end{pmatrix}, \\
 T^2S^6 &= \begin{pmatrix} 0 & \rho^5 & 0 \\ 0 & 0 & \rho^2 \\ \rho^6 & 0 & 0 \end{pmatrix}, \quad T^2S^7 = \begin{pmatrix} 0 & \rho^8 & 0 \\ 0 & 0 & \rho^{11} \\ \rho^7 & 0 & 0 \end{pmatrix}, \quad T^2S^8 = \begin{pmatrix} 0 & \rho^{11} & 0 \\ 0 & 0 & \rho^7 \\ \rho^8 & 0 & 0 \end{pmatrix}, \\
 T^2S^9 &= \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^3 \\ \rho^9 & 0 & 0 \end{pmatrix}, \quad T^2S^{10} = \begin{pmatrix} 0 & \rho^4 & 0 \\ 0 & 0 & \rho^{12} \\ \rho^{10} & 0 & 0 \end{pmatrix}, \quad T^2S^{11} = \begin{pmatrix} 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \\ \rho^{11} & 0 & 0 \end{pmatrix}, \\
 T^2S^{12} &= \begin{pmatrix} 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^4 \\ \rho^{12} & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
\mathcal{C}_4 : \quad S &= \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}, \quad S^3 = \begin{pmatrix} \rho^3 & 0 & 0 \\ 0 & \rho^9 & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad S^9 = \begin{pmatrix} \rho^9 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^3 \end{pmatrix} \\
\mathcal{C}_5 : \quad S^4 &= \begin{pmatrix} \rho^4 & 0 & 0 \\ 0 & \rho^{12} & 0 \\ 0 & 0 & \rho^{10} \end{pmatrix}, \quad S^{10} = \begin{pmatrix} \rho^{10} & 0 & 0 \\ 0 & \rho^4 & 0 \\ 0 & 0 & \rho^{12} \end{pmatrix}, \quad S^{12} = \begin{pmatrix} \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^4 \end{pmatrix} \\
\mathcal{C}_6 : \quad S^2 &= \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \end{pmatrix}, \quad S^5 = \begin{pmatrix} \rho^5 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^6 \end{pmatrix}, \quad S^6 = \begin{pmatrix} \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} \\
\mathcal{C}_7 : \quad S^7 &= \begin{pmatrix} \rho^7 & 0 & 0 \\ 0 & \rho^8 & 0 \\ 0 & 0 & \rho^{11} \end{pmatrix}, \quad S^8 = \begin{pmatrix} \rho^8 & 0 & 0 \\ 0 & \rho^{11} & 0 \\ 0 & 0 & \rho^7 \end{pmatrix}, \quad S^{11} = \begin{pmatrix} \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \end{pmatrix}
\end{aligned}$$

while for the 3-dimensional representation $\mathbf{3}_2$ the elements are

$$\begin{aligned}
\mathcal{C}_1 : \quad e &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathcal{C}_2 : \quad T &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & 0 & \rho^5 \\ \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \end{pmatrix}, \quad TS^2 = \begin{pmatrix} 0 & 0 & \rho^{10} \\ \rho^4 & 0 & 0 \\ 0 & \rho^{12} & 0 \end{pmatrix}, \\
TS^3 &= \begin{pmatrix} 0 & 0 & \rho^2 \\ \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \end{pmatrix}, \quad TS^4 = \begin{pmatrix} 0 & 0 & \rho^7 \\ \rho^8 & 0 & 0 \\ 0 & \rho^{11} & 0 \end{pmatrix}, \quad TS^5 = \begin{pmatrix} 0 & 0 & \rho^{12} \\ \rho^{10} & 0 & 0 \\ 0 & \rho^4 & 0 \end{pmatrix}, \\
TS^6 &= \begin{pmatrix} 0 & 0 & \rho^4 \\ \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \end{pmatrix}, \quad TS^7 = \begin{pmatrix} 0 & 0 & \rho^9 \\ \rho & 0 & 0 \\ 0 & \rho^3 & 0 \end{pmatrix}, \quad TS^8 = \begin{pmatrix} 0 & 0 & \rho \\ \rho^3 & 0 & 0 \\ 0 & \rho^9 & 0 \end{pmatrix}, \\
TS^9 &= \begin{pmatrix} 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \\ 0 & \rho^2 & 0 \end{pmatrix}, \quad TS^{10} = \begin{pmatrix} 0 & 0 & \rho^{11} \\ \rho^7 & 0 & 0 \\ 0 & \rho^8 & 0 \end{pmatrix}, \quad TS^{11} = \begin{pmatrix} 0 & 0 & \rho^3 \\ \rho^9 & 0 & 0 \\ 0 & \rho & 0 \end{pmatrix}, \\
TS^{12} &= \begin{pmatrix} 0 & 0 & \rho^8 \\ \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3 : \quad T^2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^2 S = \begin{pmatrix} 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \\ \rho^2 & 0 & 0 \end{pmatrix}, \quad T^2 S^2 = \begin{pmatrix} 0 & \rho^{12} & 0 \\ 0 & 0 & \rho^{10} \\ \rho^4 & 0 & 0 \end{pmatrix}, \\
T^2 S^3 &= \begin{pmatrix} 0 & \rho^5 & 0 \\ 0 & 0 & \rho^2 \\ \rho^6 & 0 & 0 \end{pmatrix}, \quad T^2 S^4 = \begin{pmatrix} 0 & \rho^{11} & 0 \\ 0 & 0 & \rho^7 \\ \rho^8 & 0 & 0 \end{pmatrix}, \quad T^2 S^5 = \begin{pmatrix} 0 & \rho^4 & 0 \\ 0 & 0 & \rho^{12} \\ \rho^{10} & 0 & 0 \end{pmatrix}, \\
T^2 S^6 &= \begin{pmatrix} 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^4 \\ \rho^{12} & 0 & 0 \end{pmatrix}, \quad T^2 S^7 = \begin{pmatrix} 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \\ \rho & 0 & 0 \end{pmatrix}, \quad T^2 S^8 = \begin{pmatrix} 0 & \rho^9 & 0 \\ 0 & 0 & \rho \\ \rho^3 & 0 & 0 \end{pmatrix}, \\
T^2 S^9 &= \begin{pmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho^6 \\ \rho^5 & 0 & 0 \end{pmatrix}, \quad T^2 S^{10} = \begin{pmatrix} 0 & \rho^8 & 0 \\ 0 & 0 & \rho^{11} \\ \rho^7 & 0 & 0 \end{pmatrix}, \quad T^2 S^{11} = \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^3 \\ \rho^9 & 0 & 0 \end{pmatrix}, \\
T^2 S^{12} &= \begin{pmatrix} 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \\ \rho^{11} & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\mathcal{C}_4 : \quad S = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \end{pmatrix}, \quad S^3 = \begin{pmatrix} \rho^6 & 0 & 0 \\ 0 & \rho^5 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}, \quad S^9 = \begin{pmatrix} \rho^5 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^6 \end{pmatrix}$$

$$\mathcal{C}_5 : \quad S^4 = \begin{pmatrix} \rho^8 & 0 & 0 \\ 0 & \rho^{11} & 0 \\ 0 & 0 & \rho^7 \end{pmatrix}, \quad S^{10} = \begin{pmatrix} \rho^7 & 0 & 0 \\ 0 & \rho^8 & 0 \\ 0 & 0 & \rho^{11} \end{pmatrix}, \quad S^{12} = \begin{pmatrix} \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \end{pmatrix}$$

$$\mathcal{C}_6 : \quad S^2 = \begin{pmatrix} \rho^4 & 0 & 0 \\ 0 & \rho^{12} & 0 \\ 0 & 0 & \rho^{10} \end{pmatrix}, \quad S^5 = \begin{pmatrix} \rho^{10} & 0 & 0 \\ 0 & \rho^4 & 0 \\ 0 & 0 & \rho^{12} \end{pmatrix}, \quad S^6 = \begin{pmatrix} \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}$$

$$\mathcal{C}_7 : \quad S^7 = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}, \quad S^8 = \begin{pmatrix} \rho^3 & 0 & 0 \\ 0 & \rho^9 & 0 \\ 0 & 0 & \rho \end{pmatrix}, \quad S^{11} = \begin{pmatrix} \rho^9 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^3 \end{pmatrix}$$

For the remaining 3-dimensional representations $\bar{\mathbf{3}}_1$ and $\bar{\mathbf{3}}_2$, the matrices representing the elements of the group can be found from those just listed for the representations $\mathbf{3}_1$ and $\mathbf{3}_2$ by performing complex conjugation. The above representation matrices can help us to see clearly how the T_{13} flavor symmetry is broken in model building. Starting from the above explicit representation matrices, we can straightforwardly get the product decomposition rules of the T_{13} group. In the following we use α_i to denote the elements of the first representation of the product and β_i to indicate those of the second representation.

$$\bullet \quad \mathbf{1}_2 \otimes \mathbf{3}_1 = \mathbf{3}_1$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega^2\alpha\beta_2 \\ \omega\alpha\beta_3 \end{pmatrix} \tag{59}$$

$$\bullet \quad \mathbf{1}_2 \otimes \bar{\mathbf{3}}_1 = \bar{\mathbf{3}}_1$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega^2\alpha\beta_2 \\ \omega\alpha\beta_3 \end{pmatrix} \tag{60}$$

- $\mathbf{1}_2 \otimes \mathbf{3}_2 = \mathbf{3}_2$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega^2\alpha\beta_2 \\ \omega\alpha\beta_3 \end{pmatrix} \quad (61)$$

- $\mathbf{1}_2 \otimes \bar{\mathbf{3}}_2 = \bar{\mathbf{3}}_2$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega^2\alpha\beta_2 \\ \omega\alpha\beta_3 \end{pmatrix} \quad (62)$$

- $\mathbf{1}_3 \otimes \mathbf{3}_1 = \mathbf{3}_1$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega\alpha\beta_2 \\ \omega^2\alpha\beta_3 \end{pmatrix} \quad (63)$$

- $\mathbf{1}_3 \otimes \bar{\mathbf{3}}_1 = \bar{\mathbf{3}}_1$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega\alpha\beta_2 \\ \omega^2\alpha\beta_3 \end{pmatrix} \quad (64)$$

- $\mathbf{1}_3 \otimes \mathbf{3}_2 = \mathbf{3}_2$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega\alpha\beta_2 \\ \omega^2\alpha\beta_3 \end{pmatrix} \quad (65)$$

- $\mathbf{1}_3 \otimes \bar{\mathbf{3}}_2 = \bar{\mathbf{3}}_2$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha\beta_1 \\ \omega\alpha\beta_2 \\ \omega^2\alpha\beta_3 \end{pmatrix} \quad (66)$$

- $\mathbf{3}_1 \otimes \mathbf{3}_1 = \bar{\mathbf{3}}_{1S} \oplus \bar{\mathbf{3}}_{1A} \oplus \mathbf{3}_2$

$$\bar{\mathbf{3}}_{1S} \sim \begin{pmatrix} \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_3\beta_1 + \alpha_1\beta_3 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix} \quad (67)$$

$$\bar{\mathbf{3}}_{1A} \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} \quad (68)$$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_2\beta_2 \\ \alpha_3\beta_3 \end{pmatrix} \quad (69)$$

- $\mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2$

$$\mathbf{1}_1 \sim \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \quad (70)$$

$$\mathbf{1}_2 \sim \alpha_1\beta_1 + \omega\alpha_2\beta_2 + \omega^2\alpha_3\beta_3 \quad (71)$$

$$\mathbf{1}_3 \sim \alpha_1\beta_1 + \omega^2\alpha_2\beta_2 + \omega\alpha_3\beta_3 \quad (72)$$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha_2 \beta_1 \\ \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \end{pmatrix} \quad (73)$$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha_1 \beta_2 \\ \alpha_2 \beta_3 \\ \alpha_3 \beta_1 \end{pmatrix} \quad (74)$$

$$\bullet \mathbf{3}_1 \otimes \mathbf{3}_2 = \mathbf{3}_1 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha_3 \beta_3 \\ \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \end{pmatrix} \quad (75)$$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \\ \alpha_2 \beta_1 \end{pmatrix} \quad (76)$$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha_3 \beta_1 \\ \alpha_1 \beta_2 \\ \alpha_2 \beta_3 \end{pmatrix} \quad (77)$$

$$\bullet \mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 = \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha_2 \beta_1 \\ \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \end{pmatrix} \quad (78)$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \alpha_3 \beta_3 \end{pmatrix} \quad (79)$$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha_2 \beta_3 \\ \alpha_3 \beta_1 \\ \alpha_1 \beta_2 \end{pmatrix} \quad (80)$$

$$\bullet \bar{\mathbf{3}}_1 \otimes \bar{\mathbf{3}}_1 = \mathbf{3}_{1S} \oplus \mathbf{3}_{1A} \oplus \bar{\mathbf{3}}_2$$

$$\mathbf{3}_{1S} \sim \begin{pmatrix} \alpha_2 \beta_3 + \alpha_3 \beta_2 \\ \alpha_3 \beta_1 + \alpha_1 \beta_3 \\ \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix} \quad (81)$$

$$\mathbf{3}_{1A} \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix} \quad (82)$$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \alpha_3 \beta_3 \end{pmatrix} \quad (83)$$

$$\bullet \bar{\mathbf{3}}_1 \otimes \mathbf{3}_2 = \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \alpha_3 \beta_3 \end{pmatrix} \quad (84)$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha_2 \beta_1 \\ \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \end{pmatrix} \quad (85)$$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha_2 \beta_3 \\ \alpha_3 \beta_1 \\ \alpha_1 \beta_2 \end{pmatrix} \quad (86)$$

$$\bullet \quad \bar{\mathbf{3}}_1 \otimes \bar{\mathbf{3}}_2 = \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha_3 \beta_3 \\ \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \end{pmatrix} \quad (87)$$

$$\mathbf{3}_2 \sim \begin{pmatrix} \alpha_3 \beta_1 \\ \alpha_1 \beta_2 \\ \alpha_2 \beta_3 \end{pmatrix} \quad (88)$$

$$\bar{\mathbf{3}}_2 \sim \begin{pmatrix} \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \\ \alpha_2 \beta_1 \end{pmatrix} \quad (89)$$

$$\bullet \quad \mathbf{3}_2 \otimes \mathbf{3}_2 = \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_{2S} \oplus \bar{\mathbf{3}}_{2A}$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha_2 \beta_2 \\ \alpha_3 \beta_3 \\ \alpha_1 \beta_1 \end{pmatrix} \quad (90)$$

$$\bar{\mathbf{3}}_{2S} \sim \begin{pmatrix} \alpha_2 \beta_3 + \alpha_3 \beta_2 \\ \alpha_3 \beta_1 + \alpha_1 \beta_3 \\ \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix} \quad (91)$$

$$\bar{\mathbf{3}}_{2A} \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix} \quad (92)$$

$$\bullet \quad \mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3 \oplus \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1$$

$$\mathbf{1}_1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3 \quad (93)$$

$$\mathbf{1}_2 \sim \alpha_1 \beta_1 + \omega \alpha_2 \beta_2 + \omega^2 \alpha_3 \beta_3 \quad (94)$$

$$\mathbf{1}_3 \sim \alpha_1 \beta_1 + \omega^2 \alpha_2 \beta_2 + \omega \alpha_3 \beta_3 \quad (95)$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha_2 \beta_3 \\ \alpha_3 \beta_1 \\ \alpha_1 \beta_2 \end{pmatrix} \quad (96)$$

$$\bar{\mathbf{3}}_1 \sim \begin{pmatrix} \alpha_3 \beta_2 \\ \alpha_1 \beta_3 \\ \alpha_2 \beta_1 \end{pmatrix} \quad (97)$$

$$\bullet \bar{\mathbf{3}}_2 \otimes \bar{\mathbf{3}}_2 = \mathbf{3}_1 \oplus \mathbf{3}_{2S} \oplus \mathbf{3}_{2A}$$

$$\mathbf{3}_1 \sim \begin{pmatrix} \alpha_2 \beta_2 \\ \alpha_3 \beta_3 \\ \alpha_1 \beta_1 \end{pmatrix} \quad (98)$$

$$\mathbf{3}_{2S} \sim \begin{pmatrix} \alpha_2 \beta_3 + \alpha_3 \beta_2 \\ \alpha_3 \beta_1 + \alpha_1 \beta_3 \\ \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix} \quad (99)$$

$$\mathbf{3}_{2A} \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix} \quad (100)$$

Appendix B: Vacuum alignment beyond leading order

In this appendix, we shall discuss the subleading corrections to the vacuum alignment induced by the higher dimensional operators. At the next level of approximation, the driving superpotential w_v in Eq.(27) is modified into $w_v + \delta w_v$. Due to the constraint imposed by $Z_4 \times Z_2$ symmetry, the correction terms are suppressed by $1/\Lambda^2$. Concretely δw_v is given by

$$\delta w_v = \frac{1}{\Lambda^2} \sum_{i=1}^{34} c_i \mathcal{O}_i^{\chi^0} + \frac{1}{\Lambda^2} \sum_{i=1}^{11} r_i \mathcal{O}_i^{\rho^0} + \frac{1}{\Lambda^2} \sum_{i=1}^{11} t_i \mathcal{O}_i^{\theta^0} + \frac{1}{\Lambda^2} \sum_{i=1}^{28} e_i \mathcal{O}_i^{\eta^0} + \frac{1}{\Lambda^2} \sum_{i=1}^{19} k_i \mathcal{O}_i^{\xi^0} \quad (101)$$

where c_i , r_i , t_i , e_i and k_i are order one coefficients, $\{\mathcal{O}_i^{\chi^0}, \mathcal{O}_i^{\rho^0}, \mathcal{O}_i^{\theta^0}, \mathcal{O}_i^{\eta^0}, \mathcal{O}_i^{\xi^0}\}$ denote the complete set of subleading contractions invariant under $T_{13} \times Z_4 \times Z_2$.

$$\begin{aligned} \mathcal{O}_1^{\chi^0} &= \chi^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\phi)_{\bar{\mathbf{3}}_2})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_2^{\chi^0} &= \chi^0((\chi\chi)_{\mathbf{3}_{1S}}(\eta\eta)_{\mathbf{3}_{2S}})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_3^{\chi^0} &= \chi^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\eta)_{\bar{\mathbf{3}}_1})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_4^{\chi^0} &= \chi^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\eta)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_5^{\chi^0} &= \chi^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\eta)_{\bar{\mathbf{3}}_2})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_6^{\chi^0} &= \chi^0((\chi\chi)_{\bar{\mathbf{3}}_2}(\phi\phi)_{\mathbf{3}_{1S}})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_7^{\chi^0} &= \chi^0((\chi\chi)_{\bar{\mathbf{3}}_2}(\eta\eta)_{\mathbf{3}_1})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_8^{\chi^0} &= \chi^0((\chi\chi)_{\bar{\mathbf{3}}_2}(\phi\eta)_{\bar{\mathbf{3}}_1})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_9^{\chi^0} &= \chi^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\phi\phi)_{\mathbf{3}_{1S}})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{10}^{\chi^0} &= \chi^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\phi\phi)_{\bar{\mathbf{3}}_2})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{11}^{\chi^0} &= \chi^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\eta\eta)_{\mathbf{3}_1})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{12}^{\chi^0} &= \chi^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\phi\eta)_{\bar{\mathbf{3}}_1})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{13}^{\chi^0} &= \chi^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\phi\eta)_{\bar{\mathbf{3}}_2})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{14}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\phi\phi)_{\mathbf{3}_{1S}})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{15}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_1})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{16}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\bar{\mathbf{3}}_{2S}}, & \mathcal{O}_{17}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\bar{\mathbf{3}}_{2A}}, & \mathcal{O}_{18}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_{2S}}, \\ \mathcal{O}_{19}^{\chi^0} &= \chi^0((\xi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_{2A}}, & \mathcal{O}_{20}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_1}(\phi\phi)_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{21}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_1}(\phi\eta)_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{22}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_2}(\phi\phi)_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{23}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_2}(\phi\eta)_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{24}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_3}(\phi\phi)_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{25}^{\chi^0} &= \chi^0(\chi\xi)_{\mathbf{1}_3}(\phi\eta)_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{26}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\phi\phi)_{\mathbf{3}_{1S}})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{27}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_1})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{28}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\bar{\mathbf{3}}_{2S}}, & \mathcal{O}_{29}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\bar{\mathbf{3}}_{2A}}, & \mathcal{O}_{30}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_{2S}}, \\ \mathcal{O}_{31}^{\chi^0} &= \chi^0((\chi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{3}_2})_{\bar{\mathbf{3}}_{2A}}, & \mathcal{O}_{32}^{\chi^0} &= \chi^0((\chi\xi)_{\bar{\mathbf{3}}_2}(\phi\phi)_{\mathbf{3}_{1S}})_{\bar{\mathbf{3}}_2}, & \mathcal{O}_{33}^{\chi^0} &= \chi^0((\chi\xi)_{\bar{\mathbf{3}}_2}(\eta\eta)_{\mathbf{3}_1})_{\bar{\mathbf{3}}_2}, \\ \mathcal{O}_{34}^{\chi^0} &= \chi^0((\chi\xi)_{\bar{\mathbf{3}}_2}(\phi\eta)_{\bar{\mathbf{3}}_1})_{\bar{\mathbf{3}}_2} \end{aligned} \quad (102)$$

$$\begin{aligned} \mathcal{O}_1^{\rho^0} &= \rho^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\eta)_{\bar{\mathbf{3}}_1})_{\mathbf{1}_3}, & \mathcal{O}_2^{\rho^0} &= \rho^0((\chi\chi)_{\bar{\mathbf{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_3}, & \mathcal{O}_3^{\rho^0} &= \rho^0((\chi\chi)_{\bar{\mathbf{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_3}, \\ \mathcal{O}_4^{\rho^0} &= \rho^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\phi\phi)_{\mathbf{3}_{1S}})_{\mathbf{1}_3}, & \mathcal{O}_5^{\rho^0} &= \rho^0((\xi\xi)_{\bar{\mathbf{3}}_{1S}}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{1}_3}, & \mathcal{O}_6^{\rho^0} &= \rho^0((\xi\xi)_{\mathbf{3}_2}(\phi\phi)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_3}, \\ \mathcal{O}_7^{\rho^0} &= \rho^0((\xi\xi)_{\mathbf{3}_2}(\phi\eta)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_3}, & \mathcal{O}_8^{\rho^0} &= \rho^0((\chi\xi)_{\mathbf{3}_2}(\phi\phi)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_3}, & \mathcal{O}_9^{\rho^0} &= \rho^0((\chi\xi)_{\mathbf{3}_2}(\phi\eta)_{\bar{\mathbf{3}}_2})_{\mathbf{1}_3}, \\ \mathcal{O}_{10}^{\rho^0} &= \rho^0((\chi\xi)_{\bar{\mathbf{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_3}, & \mathcal{O}_{11}^{\rho^0} &= \rho^0((\chi\xi)_{\bar{\mathbf{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_3} \end{aligned} \quad (103)$$

$$\begin{aligned}
\mathcal{O}_1^{\theta^0} &= \theta^0((\chi\chi)_{\mathbf{3}_{1S}}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{1}_2}, & \mathcal{O}_2^{\theta^0} &= \theta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_2}, & \mathcal{O}_3^{\theta^0} &= \theta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_2}, \\
\mathcal{O}_4^{\theta^0} &= \theta^0((\xi\xi)_{\mathbf{\bar{3}}_{1S}}(\phi\phi)_{\mathbf{3}_{1S}})_{\mathbf{1}_2}, & \mathcal{O}_5^{\theta^0} &= \theta^0((\xi\xi)_{\mathbf{\bar{3}}_{1S}}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{1}_2}, & \mathcal{O}_6^{\theta^0} &= \theta^0((\xi\xi)_{\mathbf{3}_2}(\phi\phi)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_2}, \\
\mathcal{O}_7^{\theta^0} &= \theta^0((\xi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_2}, & \mathcal{O}_8^{\theta^0} &= \theta^0((\chi\xi)_{\mathbf{3}_2}(\phi\phi)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_2}, & \mathcal{O}_9^{\theta^0} &= \theta^0((\chi\xi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_2}, \\
\mathcal{O}_{10}^{\theta^0} &= \theta^0((\chi\xi)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_2}, & \mathcal{O}_{11}^{\theta^0} &= \theta^0((\chi\xi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_2}
\end{aligned} \tag{104}$$

$$\begin{aligned}
\mathcal{O}_1^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{3}_{1S}}(\phi\phi)_{\mathbf{3}_{1S}})_{\mathbf{3}_2}, & \mathcal{O}_2^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{\bar{3}}_2}(\phi\phi)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2S}}, & \mathcal{O}_3^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{3}_{1S}}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{3}_2}, \\
\mathcal{O}_4^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{3}_{1S}}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{3}_2}, & \mathcal{O}_5^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{3}_2}, & \mathcal{O}_6^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2S}}, \\
\mathcal{O}_7^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2A}}, & \mathcal{O}_8^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{3}_{1S}}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{3}_2}, & \mathcal{O}_9^{\eta^0} &= \eta^0((\phi\phi)_{\mathbf{3}_{1S}}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{3}_2}, \\
\mathcal{O}_{10}^{\eta^0} &= \eta^0((\eta\eta)_{\mathbf{3}_1}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{3}_2}, & \mathcal{O}_{11}^{\eta^0} &= \eta^0((\eta\eta)_{\mathbf{3}_1}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{3}_2}, & \mathcal{O}_{12}^{\eta^0} &= \eta^0((\eta\eta)_{\mathbf{3}_{2S}}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{3}_2}, \\
\mathcal{O}_{13}^{\eta^0} &= \eta^0((\eta\eta)_{\mathbf{3}_1}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{3}_2}, & \mathcal{O}_{14}^{\eta^0} &= \eta^0((\eta\eta)_{\mathbf{3}_1}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{3}_2}, & \mathcal{O}_{15}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{3}_{1S}}(\chi\chi)_{\mathbf{3}_{1S}})_{\mathbf{3}_2}, \\
\mathcal{O}_{16}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\chi\chi)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2S}}, & \mathcal{O}_{17}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{3}_{1S}}(\chi\xi)_{\mathbf{3}_2})_{\mathbf{3}_2}, & \mathcal{O}_{18}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\chi\xi)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2S}}, \\
\mathcal{O}_{19}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\chi\xi)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_{2A}}, & \mathcal{O}_{20}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{3}_{1S}}(\xi\xi)_{\mathbf{\bar{3}}_{1S}})_{\mathbf{3}_2}, & \mathcal{O}_{21}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{3}_{1S}}(\xi\xi)_{\mathbf{3}_2})_{\mathbf{3}_2}, \\
\mathcal{O}_{22}^{\eta^0} &= \eta^0((\chi\chi)_{\mathbf{\bar{3}}_2}(\xi\xi)_{\mathbf{\bar{3}}_{1S}})_{\mathbf{3}_2}, & \mathcal{O}_{23}^{\eta^0} &= \eta^0((\xi\xi)_{\mathbf{\bar{3}}_{1S}}(\chi\xi)_{\mathbf{3}_2})_{\mathbf{3}_2}, & \mathcal{O}_{24}^{\eta^0} &= \eta^0((\xi\xi)_{\mathbf{\bar{3}}_{1S}}(\chi\xi)_{\mathbf{\bar{3}}_2})_{\mathbf{3}_2}, \\
\mathcal{O}_{25}^{\eta^0} &= \eta^0(\xi\xi)_{\mathbf{3}_2}(\chi\xi)_{\mathbf{1}_1}, & \mathcal{O}_{26}^{\eta^0} &= \eta^0(\xi\xi)_{\mathbf{3}_2}(\chi\xi)_{\mathbf{1}_2}, & \mathcal{O}_{27}^{\eta^0} &= \eta^0(\xi\xi)_{\mathbf{3}_2}(\chi\xi)_{\mathbf{1}_3}, \\
\mathcal{O}_{28}^{\eta^0} &= \eta^0((\xi\xi)_{\mathbf{\bar{3}}_{1S}}(\xi\xi)_{\mathbf{3}_2})_{\mathbf{3}_2}
\end{aligned} \tag{105}$$

$$\begin{aligned}
\mathcal{O}_1^{\xi^0} &= \xi^0((\chi\phi)_{\mathbf{3}_{1S}}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{1}_1}, & \mathcal{O}_2^{\xi^0} &= \xi^0((\chi\phi)_{\mathbf{3}_{1A}}(\phi\eta)_{\mathbf{\bar{3}}_1})_{\mathbf{1}_1}, & \mathcal{O}_3^{\xi^0} &= \xi^0((\chi\phi)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_1}, \\
\mathcal{O}_4^{\xi^0} &= \xi^0((\chi\phi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_1}, & \mathcal{O}_5^{\xi^0} &= \xi^0((\chi\eta)_{\mathbf{\bar{3}}_1}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{1}_1}, & \mathcal{O}_6^{\xi^0} &= \xi^0((\chi\eta)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_1}, \\
\mathcal{O}_7^{\xi^0} &= \xi^0((\xi\phi)_{\mathbf{3}_2}(\phi\phi)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_1}, & \mathcal{O}_8^{\xi^0} &= \xi^0((\xi\phi)_{\mathbf{3}_2}(\phi\eta)_{\mathbf{\bar{3}}_2})_{\mathbf{1}_1}, & \mathcal{O}_9^{\xi^0} &= \xi^0((\xi\phi)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_1}, \\
\mathcal{O}_{10}^{\xi^0} &= \xi^0((\xi\phi)_{\mathbf{\bar{3}}_2}(\phi\eta)_{\mathbf{3}_2})_{\mathbf{1}_1}, & \mathcal{O}_{11}^{\xi^0} &= \xi^0((\xi\eta)_{\mathbf{\bar{3}}_1}(\eta\eta)_{\mathbf{3}_1})_{\mathbf{1}_1}, & \mathcal{O}_{12}^{\xi^0} &= \xi^0((\xi\eta)_{\mathbf{\bar{3}}_2}(\eta\eta)_{\mathbf{3}_{2S}})_{\mathbf{1}_1}
\end{aligned} \tag{106}$$

The subleading contribution δw_v modifies the LO VEVs, then the new vacuum configuration can be parameterized as

$$\begin{aligned}
\langle\chi\rangle &= \begin{pmatrix} v_\chi + \delta v_{\chi_1} \\ v_\chi + \delta v_{\chi_2} \\ v_\chi + \delta v_{\chi_3} \end{pmatrix}, & \langle\xi\rangle &= \begin{pmatrix} v_\xi + \delta v_{\xi_1} \\ v_\xi + \delta v_{\xi_2} \\ v_\xi \end{pmatrix}, \\
\langle\phi\rangle &= \begin{pmatrix} \delta v_{\phi_1} \\ v_\phi + \delta v_{\phi_2} \\ -v_\phi \end{pmatrix}, & \langle\eta\rangle &= \begin{pmatrix} \delta v_{\eta_1} \\ v_\eta \\ \delta v_{\eta_3} \end{pmatrix}
\end{aligned} \tag{107}$$

where the shifts δv_{ξ_3} , δv_{ϕ_3} and δv_{η_2} have been absorbed into the undetermined parameters v_ξ , v_ϕ and v_η . Similar to section 4, the new vacua is obtained by searching for the zeros of the F-terms, i.e. the first derivative of $w_v + \delta w_v$ with respect to the driving fields χ^0 , ρ^0 , θ^0 , η^0 and ξ^0 . By keeping only the terms linear in the shift δv and neglecting the terms proportional to $\delta v/\Lambda$, the minimization equations become

$$\begin{aligned}
2f_1 v_\chi \delta v_{\chi_1} + f_2 v_\xi \delta v_{\chi_2} + f_2 v_\chi \delta v_{\xi_1} + a_1 v_\chi v_\xi v_\phi^2 / \Lambda^2 &= 0 \\
2f_1 v_\chi \delta v_{\chi_2} + f_2 v_\xi \delta v_{\chi_3} + f_2 v_\chi \delta v_{\xi_2} + a_2 v_\chi v_\xi v_\phi^2 / \Lambda^2 &= 0 \\
2f_1 v_\chi \delta v_{\chi_3} + f_2 v_\xi \delta v_{\chi_1} + a_3 v_\chi v_\xi v_\phi^2 / \Lambda^2 &= 0 \\
f_3 [v_\xi (\delta v_{\chi_1} + \omega^2 \delta v_{\chi_2} + \omega \delta v_{\chi_3}) + v_\chi (\delta v_{\xi_1} + \omega^2 \delta v_{\xi_2})] + a_4 v_\chi v_\xi v_\phi^2 / \Lambda^2 &= 0 \\
f_4 [v_\xi (\delta v_{\chi_1} + \omega \delta v_{\chi_2} + \omega^2 \delta v_{\chi_3}) + v_\chi (\delta v_{\xi_1} + \omega \delta v_{\xi_2})] + a_5 v_\chi v_\xi v_\phi^2 / \Lambda^2 &= 0
\end{aligned} \tag{108}$$

where the coefficients $a_i (i = 1 - 5)$ are linear combinations of the subleading coefficients

$$\begin{aligned}
a_1 &= 2c_1 v_\chi / v_\xi + c_8 v_\chi v_\eta / (v_\xi v_\phi) + (-4c_9 + 2c_{10}) v_\xi / v_\chi + 2c_{11} v_\xi v_\eta^2 / (v_\chi v_\phi^2) + (-3c_{21} + c_{34}) v_\eta / v_\phi \\
a_2 &= 2(c_3 - c_5) v_\chi v_\eta / (v_\xi v_\phi) + 2(c_{10} - c_{14}) v_\xi / v_\phi + c_{15} v_\xi v_\eta^2 / (v_\chi v_\phi^2) + 3(c_{20} - c_{26}) + c_{27} v_\eta^2 / v_\phi^2 \\
a_3 &= 2(c_1 - c_6) v_\chi / v_\xi + c_7 v_\chi v_\eta^2 / (v_\xi v_\phi^2) + 2(c_{12} - c_{13}) v_\xi v_\eta / (v_\chi v_\phi) + 3c_{20} - 2c_{32} + c_{33} v_\eta^2 / v_\phi^2 \\
a_4 &= 2\omega r_1 v_\chi v_\eta / (v_\xi v_\phi) - (4r_4 + r_6) v_\xi / v_\chi + 2r_5 v_\xi v_\eta^2 / (v_\chi v_\phi^2) - r_7 v_\xi v_\eta / (v_\chi v_\phi) - r_8 - r_9 v_\eta / v_\phi \\
a_5 &= 2\omega^2 t_1 v_\chi v_\eta / (v_\xi v_\phi) - (4t_4 + t_6) v_\xi / v_\chi + 2t_5 v_\xi v_\eta^2 / (v_\chi v_\phi^2) - t_7 v_\xi v_\eta / (v_\chi v_\phi) - t_8 - t_9 v_\eta / v_\phi
\end{aligned} \tag{109}$$

The equations Eq.(108) are linear in $\delta v_{\chi_i} (i = 1, 2, 3)$ and $\delta v_{\xi_i} (i = 1, 2)$, and can be solved straightforwardly by

$$\begin{aligned}
\frac{\delta v_{\chi_1}}{v_\chi} &= (6a_1 + 2a_2 + 5a_3) \frac{v_\phi^2}{13f_2\Lambda^2} - 2(7 + 5\omega)a_4 \frac{v_\phi^2}{39f_3\Lambda^2} - 2(2 - 5\omega)a_5 \frac{v_\phi^2}{39f_4\Lambda^2} \\
\frac{\delta v_{\chi_2}}{v_\chi} &= (2a_1 + 5a_2 + 6a_3) \frac{v_\phi^2}{13f_2\Lambda^2} - (3 + 4\omega)a_4 \frac{v_\phi^2}{13f_3\Lambda^2} + (1 + 4\omega)a_5 \frac{v_\phi^2}{13f_4\Lambda^2} \\
\frac{\delta v_{\chi_3}}{v_\chi} &= (3a_1 + a_2 + 9a_3) \frac{v_\phi^2}{13f_2\Lambda^2} - (7 + 5\omega)a_4 \frac{v_\phi^2}{39f_3\Lambda^2} - (2 - 5\omega)a_5 \frac{v_\phi^2}{39f_4\Lambda^2} \\
\frac{\delta v_{\xi_1}}{v_\xi} &= -(3a_1 + a_2 - 4a_3) \frac{v_\phi^2}{13f_2\Lambda^2} - (19 + 8\omega)a_4 \frac{v_\phi^2}{39f_3\Lambda^2} - (11 - 8\omega)a_5 \frac{v_\phi^2}{39f_4\Lambda^2} \\
\frac{\delta v_{\xi_2}}{v_\xi} &= (a_1 - 4a_2 + 3a_3) \frac{v_\phi^2}{13f_2\Lambda^2} - (11 + 19\omega)a_4 \frac{v_\phi^2}{39f_3\Lambda^2} + (8 + 19\omega)a_5 \frac{v_\phi^2}{39f_4\Lambda^2} \tag{110}
\end{aligned}$$

From the above equations, we clearly see that all the shifts $\delta v_{\chi_1}/v_\chi$, $\delta v_{\chi_2}/v_\chi$, $\delta v_{\chi_3}/v_\chi$, $\delta v_{\xi_1}/v_\xi$ and $\delta v_{\xi_2}/v_\xi$ are of order ε'^2 . The minimization equations for δv_{ϕ_1} , δv_{ϕ_2} , δv_{η_1} and δv_{η_3} are

$$\begin{aligned}
2g_1 v_\eta \delta v_{\eta_3} - g_2 v_\phi \delta v_{\eta_1} + b_1 v_\eta v_\phi^3 / \Lambda^2 &= 0 \\
g_2 v_\eta \delta v_{\phi_1} + b_2 v_\eta v_\phi^3 / \Lambda^2 &= 0 \\
2g_1 v_\eta \delta v_{\eta_1} + g_2 v_\phi \delta v_{\eta_3} + b_3 v_\eta v_\phi^3 / \Lambda^2 &= 0 \\
h(v_\xi \delta v_{\phi_1} + v_\xi \delta v_{\phi_2} + v_\phi \delta v_{\xi_2}) + b_4 v_\xi v_\phi^3 / \Lambda^2 &= 0 \tag{111}
\end{aligned}$$

where the coefficients $b_i (i = 1 - 4)$ are given by

$$\begin{aligned}
d &= [2(2e_{15} + e_{16})v_\chi^4 + 2(e_{17} + e_{18})v_\chi^3 v_\xi + 2(2e_{20} + e_{21} + e_{22})v_\chi^2 v_\xi^2 + (2e_{23} + 2e_{24} + 3e_{25})v_\chi v_\xi^3 \\
&\quad + 2e_{28}v_\xi^4] / (v_\eta v_\phi^3) \\
b_1 &= (4e_1 + 2e_2)v_\phi / v_\eta - 2e_8 v_\eta / v_\phi + e_{13} v_\eta^3 / v_\phi^3 + d \\
b_2 &= -e_6 - e_7 + d \\
b_3 &= -2e_3 - e_6 + e_7 + e_{10} v_\eta^2 / v_\phi^2 + d \\
b_4 &= (k_1 + k_2)v_\chi v_\eta / (v_\xi v_\phi) \tag{112}
\end{aligned}$$

The solutions to Eq.(111) are given by

$$\begin{aligned}
\frac{\delta v_{\phi_1}}{v_\phi} &= -\frac{b_2}{g_2} \frac{v_\phi^2}{\Lambda^2} \\
\frac{\delta v_{\phi_2}}{v_\phi} &= \left(\frac{b_2}{g_2} - \frac{b_4}{h}\right) \frac{v_\phi^2}{\Lambda^2} - \frac{\delta v_{\xi_2}}{v_\xi} \\
\frac{\delta v_{\eta_1}}{v_\eta} &= \frac{(b_1 g_2 v_\phi - 2 b_3 g_1 v_\eta) v_\phi}{4 g_1^2 v_\eta^2 + g_2^2 v_\phi^2} \frac{v_\phi^2}{\Lambda^2} \\
\frac{\delta v_{\eta_3}}{v_\eta} &= -\frac{(2 b_1 g_1 v_\eta + b_3 g_2 v_\phi) v_\phi}{4 g_1^2 v_\eta^2 + g_2^2 v_\phi^2} \frac{v_\phi^2}{\Lambda^2}
\end{aligned} \tag{113}$$

Obviously $\delta v_{\phi_1}/v_\phi$, $\delta v_{\phi_2}/v_\phi$, $\delta v_{\eta_1}/v_\eta$ and $\delta v_{\eta_3}/v_\eta$ are of order ε'^2 as well. As is shown in Eq.(105), the subleading terms proportional to η^0 are of the structures $\eta^0 \Phi_\nu^4$ or $\eta^0 \Phi_\ell^4$, the contributions of the latter operator to the vacuum alignment are parameterized in terms of the parameter d in Eq.(112). If we have a large VEV of Φ_ν with $\langle \Phi_\nu \rangle / \Lambda \sim \lambda_c$, then the structure $\eta^0 \Phi_\nu^4$ is dominant. On the other hand, if the VEVs of Φ_ν and Φ_ℓ are of the same order of magnitude, the contributions of the two type of operators are comparable.

References

- [1] T. Schwetz, M. Tortola and J. W. F. Valle, arXiv:1103.0734 [hep-ph]; T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. **10**, 113011 (2008) [arXiv:0808.2016 [hep-ph]]; M. Maltoni and T. Schwetz, arXiv:0812.3161 [hep-ph].
- [2] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A. M. Rotunno, arXiv:0809.2936 [hep-ph]; G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A. M. Rotunno, Phys. Rev. Lett. **101** (2008) 141801 [arXiv:0806.2649 [hep-ph]].
- [3] M. C. Gonzalez-Garcia, M. Maltoni and J. Salvado, JHEP **1004**, 056 (2010) [arXiv:1001.4524 [hep-ph]].
- [4] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **530**, 167 (2002), hep-ph/0202074; P. F. Harrison and W. G. Scott, Phys. Lett. B **535**, 163 (2002), hep-ph/0203209; Z. Z. Xing, Phys. Lett. B **533**, 85 (2002), hep-ph/0204049; X. G. He and A. Zee, Phys. Lett. B **560**, 87 (2003), hep-ph/0301092.
- [5] E. Ma and G. Rajasekaran, Phys. Rev. D **64** (2001) 113012, arXiv:hep-ph/0106291; K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B **552** (2003) 207, arXiv:hep-ph/0206292; M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, arXiv:hep-ph/0312244; Phys. Rev. D **69** (2004) 093006 [arXiv:hep-ph/0312265]; E. Ma, Phys. Rev. D **70** (2004) 031901; Phys. Rev. D **70** (2004) 031901 [arXiv:hep-ph/0404199]; G. Altarelli and F. Feruglio, Nucl. Phys. B **720**, 64 (2005), hep-ph/0504165; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B **724** (2005) 423 [arXiv:hep-ph/0504181]; M. Hirsch, A. Villanova del Moral, J. W. F. Valle and E. Ma, Phys. Rev. D **72** (2005) 091301 [Erratum-ibid. D **72** (2005) 119904] [arXiv:hep-ph/0507148]. K. S. Babu and X. G. He, arXiv:hep-ph/0507217; A. Zee, Phys. Lett. B **630** (2005) 58 [arXiv:hep-ph/0508278]; E. Ma, Phys. Rev. D **73** (2006) 057304

- [arXiv:hep-ph/0511133]; G. Altarelli and F. Feruglio, Nucl. Phys. B **741**, 215 (2006), hep-ph/0512103; X. G. He, Y. Y. Keum and R. R. Volkas, JHEP **0604** (2006) 039 [arXiv:hep-ph/0601001]; E. Ma, H. Sawanaka and M. Tanimoto, Phys. Lett. B **641**, 301 (2006), hep-ph/0606103; S. F. King and M. Malinsky, Phys. Lett. B **645** (2007) 351 [arXiv:hep-ph/0610250]; M. Hirsch, A. S. Joshipura, S. Kaneko and J. W. F. Valle, Phys. Rev. Lett. **99**, 151802 (2007) [arXiv:hep-ph/0703046]. F. Bazzocchi, S. Kaneko and S. Morisi, JHEP **0803** (2008) 063 [arXiv:0707.3032 [hep-ph]]. F. Bazzocchi, S. Morisi and M. Picariello, Phys. Lett. B **659** (2008) 628 [arXiv:0710.2928 [hep-ph]]; M. Honda and M. Tanimoto, Prog. Theor. Phys. **119** (2008) 583 [arXiv:0801.0181 [hep-ph]]; M. Hirsch, S. Morisi and J. W. F. Valle, Phys. Rev. D **78** (2008) 093007 [arXiv:0804.1521 [hep-ph]]; Y. Lin, Nucl. Phys. B **813**, 91 (2009) [arXiv:0804.2867 [hep-ph]]; P. H. Frampton and S. Matsuzaki, arXiv:0806.4592 [hep-ph]; S. Morisi, Phys. Rev. D **79**, 033008 (2009) [arXiv:0901.1080 [hep-ph]]; M. C. Chen and S. F. King, JHEP **0906**, 072 (2009) [arXiv:0903.0125 [hep-ph]]; G. Altarelli and D. Meloni, J. Phys. G **36**, 085005 (2009) [arXiv:0905.0620 [hep-ph]]; Y. Lin, Nucl. Phys. B **824**, 95 (2010) [arXiv:0905.3534 [hep-ph]]; F. Feruglio, C. Hagedorn and L. Merlo, JHEP **1003**, 084 (2010) [arXiv:0910.4058 [hep-ph]]; S. Morisi and E. Peinado, Phys. Rev. D **80**, 113011 (2009) [arXiv:0910.4389 [hep-ph]]; J. Berger and Y. Grossman, JHEP **1002**, 071 (2010) [arXiv:0910.4392 [hep-ph]]; Y. Lin, L. Merlo and A. Paris, Nucl. Phys. B **835**, 238 (2010) [arXiv:0911.3037 [hep-ph]]; Y. H. Ahn and C. S. Chen, Phys. Rev. D **81**, 105013 (2010) [arXiv:1001.2869 [hep-ph]]; J. Barry and W. Rodejohann, Phys. Rev. D **81**, 093002 (2010) [arXiv:1003.2385 [hep-ph]]; Y. H. Ahn, H. Y. Cheng and S. Oh, Phys. Rev. D **83**, 076012 (2011) [arXiv:1102.0879 [hep-ph]].
- [6] C. Csaki, C. Delaunay, C. Grojean and Y. Grossman, JHEP **0810**, 055 (2008) [arXiv:0806.0356 [hep-ph]]; F. del Aguila, A. Carmona and J. Santiago, arXiv:1001.5151 [hep-ph]; A. Kadosh and E. Pallante, arXiv:1004.0321 [hep-ph].
- [7] (SU(5) models with A_4 symmetry) G. Altarelli, F. Feruglio and C. Hagedorn, JHEP **0803**, 052 (2008) [arXiv:0802.0090 [hep-ph]]; P. Ciafaloni, M. Picariello, E. Torrente-Lujan and A. Urbano, Phys. Rev. D **79**, 116010 (2009) [arXiv:0901.2236 [hep-ph]]; T. J. Burrows and S. F. King, arXiv:0909.1433 [hep-ph]; P. Ciafaloni, M. Picariello, A. Urbano and E. Torrente-Lujan, Phys. Rev. D **81**, 016004 (2010) [arXiv:0909.2553 [hep-ph]]; I. K. Cooper, S. F. King and C. Luhn, arXiv:1004.3243 [hep-ph]; S. Antusch, S. F. King and M. Spinrath, Phys. Rev. D **83**, 013005 (2011) [arXiv:1005.0708 [hep-ph]].
- [8] (SO(10) models with A_4 symmetry) S. Morisi, M. Picariello and E. Torrente-Lujan, Phys. Rev. D **75**, 075015 (2007) [arXiv:hep-ph/0702034]; W. Grimus and H. Kuhbock, Phys. Rev. D **77**, 055008 (2008) [arXiv:0710.1585 [hep-ph]]; F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, J. Phys. G **36** (2009) 015002 [arXiv:0802.1693 [hep-ph]]; F. Bazzocchi, M. Frigerio and S. Morisi, Phys. Rev. D **78**, 116018 (2008) [arXiv:0809.3573 [hep-ph]]; A. Albaid, Phys. Rev. D **80**, 093002 (2009) [arXiv:0909.1762 [hep-ph]].
- [9] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B **809**, 218 (2009) [arXiv:0807.3160 [hep-ph]]; C. Hagedorn, E. Molinaro and S. T. Petcov, JHEP **1002**, 047 (2010) [arXiv:0911.3605 [hep-ph]]; F. Feruglio, C. Hagedorn, Y. Lin and

- L. Merlo, arXiv:0911.3874 [hep-ph]; G. J. Ding and J. F. Liu, JHEP **1005**, 029 (2010) [arXiv:0911.4799 [hep-ph]].
- [10] G. C. Branco, R. Gonzalez Felipe, M. N. Rebelo and H. Serodio, Phys. Rev. D **79**, 093008 (2009) [arXiv:0904.3076 [hep-ph]]; E. Bertuzzo, P. Di Bari, F. Feruglio and E. Nardi, JHEP **0911**, 036 (2009) [arXiv:0908.0161 [hep-ph]]; C. Hagedorn, E. Molinaro and S. T. Petcov, JHEP **0909**, 115 (2009) [arXiv:0908.0240 [hep-ph]]; D. Aristizabal Sierra, F. Bazzocchi, I. de Medeiros Varzielas, L. Merlo and S. Morisi, Nucl. Phys. B **827**, 34 (2010) [arXiv:0908.0907 [hep-ph]]; R. G. Felipe and H. Serodio, Phys. Rev. D **81**, 053008 (2010) [arXiv:0908.2947 [hep-ph]].
- [11] C. Luhn, S. Nasri and P. Ramond, Phys. Lett. B **652**, 27 (2007) [arXiv:0706.2341 [hep-ph]]; C. Hagedorn, M. A. Schmidt and A. Y. Smirnov, Phys. Rev. D **79**, 036002 (2009) [arXiv:0811.2955 [hep-ph]]; S. F. King and C. Luhn, JHEP **0910**, 093 (2009) [arXiv:0908.1897 [hep-ph]].
- [12] A. Aranda, C. D. Carone and R. F. Lebed, Phys. Lett. B **474**, 170 (2000) [arXiv:hep-ph/9910392]; A. Aranda, C. D. Carone and R. F. Lebed, Phys. Rev. D **62**, 016009 (2000) [arXiv:hep-ph/0002044]; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B **775**, 120 (2007), hep-ph/0702194; G. J. Ding, Phys. Rev. D **78**, 036011 (2008) [arXiv:0803.2278 [hep-ph]]; M. C. Chen and K. T. Mahanthappa, Phys. Lett. B **652**, 34 (2007), arXiv:0705.0714 [hep-ph]; P. H. Frampton and T. W. Kephart, JHEP **0709**, 110 (2007), arXiv:0706.1186 [hep-ph]; A. Aranda, Phys. Rev. D **76**, 111301 (2007), arXiv:0707.3661 [hep-ph]; P. H. Frampton, T. W. Kephart and S. Matsuzaki, Phys. Rev. D **78**, 073004 (2008) [arXiv:0807.4713 [hep-ph]]; Y. BenTov and A. Zee, arXiv:1101.1987 [hep-ph].
- [13] C. Hagedorn, M. Lindner and R. N. Mohapatra, JHEP **0606**, 042 (2006) [arXiv:hep-ph/0602244]; E. Ma, Phys. Lett. B **632**, 352 (2006) [arXiv:hep-ph/0508231]; F. Bazzocchi and S. Morisi, arXiv:0811.0345 [hep-ph]; H. Ishimori, Y. Shimizu and M. Tanimoto, Prog. Theor. Phys. **121**, 769 (2009) [arXiv:0812.5031 [hep-ph]]; F. Bazzocchi, L. Merlo and S. Morisi, Nucl. Phys. B **816**, 204 (2009) [arXiv:0901.2086 [hep-ph]]; F. Bazzocchi, L. Merlo and S. Morisi, Phys. Rev. D **80**, 053003 (2009) [arXiv:0902.2849 [hep-ph]]; G. Altarelli, F. Feruglio and L. Merlo, JHEP **0905**, 020 (2009) [arXiv:0903.1940 [hep-ph]]; W. Grimus, L. Lavoura and P. O. Ludl, J. Phys. G **36**, 115007 (2009) [arXiv:0906.2689 [hep-ph]]; G. J. Ding, Nucl. Phys. B **827**, 82 (2010) [arXiv:0909.2210 [hep-ph]]; B. Dutta, Y. Mimura and R. N. Mohapatra, JHEP **1005**, 034 (2010) [arXiv:0911.2242 [hep-ph]]; D. Meloni, J. Phys. G **37**, 055201 (2010) [arXiv:0911.3591 [hep-ph]]; S. Morisi and E. Peinado, Phys. Rev. D **81**, 085015 (2010) [arXiv:1001.2265 [hep-ph]]; C. Hagedorn, S. F. King and C. Luhn, arXiv:1003.4249 [hep-ph]; R. d. A. Toorop, F. Bazzocchi and L. Merlo, arXiv:1003.4502 [hep-ph]; Y. H. Ahn, S. K. Kang, C. S. Kim and T. P. Nguyen, Phys. Rev. D **82**, 093005 (2010) [arXiv:1004.3469 [hep-ph]]; H. Ishimori, K. Saga, Y. Shimizu and M. Tanimoto, arXiv:1004.5004 [hep-ph]; G. J. Ding, Nucl. Phys. B **846**, 394 (2011) [arXiv:1006.4800 [hep-ph]]; K. M. Patel, Phys. Lett. B **695**, 225 (2011) [arXiv:1008.5061 [hep-ph]]; H. Ishimori, Y. Shimizu, M. Tanimoto and A. Watanabe, Phys. Rev. D **83**, 033004

- (2011) [arXiv:1010.3805 [hep-ph]]; G. J. Ding and D. M. Pan, arXiv:1011.5306 [hep-ph]; R. Z. Yang and H. Zhang, arXiv:1104.0380 [hep-ph].
- [14] C. S. Lam, Phys. Rev. Lett. **101**, 121602 (2008) [arXiv:0804.2622 [hep-ph]]; C. S. Lam, Phys. Rev. D **78**, 073015 (2008) [arXiv:0809.1185 [hep-ph]]; C. S. Lam, arXiv:0907.2206 [hep-ph].
- [15] I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B **648**, 201 (2007) [arXiv:hep-ph/0607045]; E. Ma, Phys. Lett. B **660**, 505 (2008) [arXiv:0709.0507 [hep-ph]]; W. Grimus and L. Lavoura, JHEP **0809**, 106 (2008) [arXiv:0809.0226 [hep-ph]]; F. Bazzocchi and I. de Medeiros Varzielas, Phys. Rev. D **79**, 093001 (2009) [arXiv:0902.3250 [hep-ph]].
- [16] S. F. King, JHEP **0508**, 105 (2005) [arXiv:hep-ph/0506297]; S. F. King and M. Malinsky, JHEP **0611**, 071 (2006) [arXiv:hep-ph/0608021].
- [17] S. F. King and G. G. Ross, Phys. Lett. B **574**, 239 (2003) [arXiv:hep-ph/0307190]; I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B **733**, 31 (2006) [arXiv:hep-ph/0507176].
- [18] L. L. Everett and A. J. Stuart, Phys. Rev. D **79**, 085005 (2009) [arXiv:0812.1057 [hep-ph]]; F. Feruglio and A. Paris, JHEP **1103**, 101 (2011) [arXiv:1101.0393 [hep-ph]].
- [19] J. A. Escobar, arXiv:1102.1649 [hep-ph].
- [20] E. Ma, Phys. Lett. B **649**, 287 (2007) [arXiv:hep-ph/0612022]; E. Ma, Europhys. Lett. **79**, 61001 (2007) [arXiv:hep-ph/0701016]; C. Hagedorn, M. A. Schmidt and A. Y. Smirnov, Phys. Rev. D **79**, 036002 (2009) [arXiv:0811.2955 [hep-ph]].
- [21] S. F. King and C. Luhn, Nucl. Phys. B **820**, 269 (2009) [arXiv:0905.1686 [hep-ph]]; S. F. King and C. Luhn, Nucl. Phys. B **832**, 414 (2010) [arXiv:0912.1344 [hep-ph]].
- [22] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211 [hep-ph]].
- [23] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, Prog. Theor. Phys. Suppl. **183**, 1 (2010) [arXiv:1003.3552 [hep-th]].
- [24] K. M. Parattu and A. Wingerter, arXiv:1012.2842 [hep-ph].
- [25] Y. Kajiyama and H. Okada, Nucl. Phys. B **848**, 303 (2011) [arXiv:1011.5753 [hep-ph]].
- [26] W. M. Fairbairn and T. Fulton, J. Math. Phys. **23**, 1747 (1982).
- [27] A.D.Thomas and G.V.Wood, Group Tables, Shiva Publishing Limited.
- [28] J. Barry and W. Rodejohann, Nucl. Phys. B **842**, 33 (2011) [arXiv:1007.5217 [hep-ph]].
- [29] F. Bazzocchi, L. Merlo and S. Morisi, Nucl. Phys. B **816**, 204 (2009) [arXiv:0901.2086 [hep-ph]]; F. Bazzocchi, L. Merlo and S. Morisi, Phys. Rev. D **80**, 053003 (2009) [arXiv:0902.2849 [hep-ph]].

- [30] G. J. Ding, Nucl. Phys. B **827**, 82 (2010) [arXiv:0909.2210 [hep-ph]].
- [31] G. J. Ding, Nucl. Phys. B **846**, 394 (2011) [arXiv:1006.4800 [hep-ph]].
- [32] G. Altarelli and F. Feruglio, Nucl. Phys. B **720**, 64 (2005), hep-ph/0504165.
- [33] G. Altarelli and F. Feruglio, Nucl. Phys. B **741**, 215 (2006), hep-ph/0512103.
- [34] Y. Lin, Nucl. Phys. B **824**, 95 (2010) [arXiv:0905.3534 [hep-ph]].
- [35] G. Altarelli, F. Feruglio and L. Merlo, JHEP **0905**, 020 (2009) [arXiv:0903.1940 [hep-ph]].
- [36] F. Ardellier *et al.* [Double Chooz Collaboration], arXiv:hep-ex/0606025.
- [37] Y. f. Wang, arXiv:hep-ex/0610024.
- [38] A. Osipowicz *et al.* [KATRIN Collaboration], arXiv:hep-ex/0109033; see also: <http://www-ik.fzk.de/~katrin/index.html>
- [39] A. Giuliani [CUORE Collaboration], J. Phys. Conf. Ser. **120** (2008) 052051.
- [40] Majorana Collaboration, arXiv:0811.2446 [nucl-ex].
- [41] A. A. Smolnikov and f. t. G. Collaboration, arXiv:0812.4194 [nucl-ex].
- [42] G. L. Fogli *et al.*, Phys. Rev. D **75**, 053001 (2007) [arXiv:hep-ph/0608060]; G. L. Fogli *et al.*, Phys. Rev. D **78**, 033010 (2008) [arXiv:0805.2517 [hep-ph]].
- [43] WMAP Collaboration, E. Komatsu *et al.*, arXiv:0803.0547 [astro-ph]; ACBAR Collaboration, C. L. Reichardt *et al.*, arXiv:0801.1491 [astro-ph]; VSA Collaboration, C. Dickinson *et al.*, Mon. Not. Roy. Astron. Soc. **353**, 732 (2004) [arXiv:astro-ph/0402498]; CBI Collaboration, A. C. S. Readhead *et al.*, Astrophys. J. **609**, 498 (2004) [arXiv:astro-ph/0402359]; BOOMERANG Collaboration, C. J. MacTavish *et al.*, Astrophys. J. **647**, 799 (2006) [arXiv:astro-ph/0507503]; SDSS Collaboration, M. Tegmark *et al.*, Phys. Rev. D **74** (2006) 123507 [arXiv:astro-ph/0608632]; SNLS Collaboration, P. Astier *et al.* Astron. Astrophys. **447**, 31 (2006) [arXiv:astro-ph/0510447]; SDSS Collaboration, D. J. Eisenstein *et al.*, Astrophys. J. **633**, 560 (2005)[arXiv:astro-ph/0501171].
- [44] P. McDonald *et al.*, Astrophys. J. Suppl. **163**, 80 (2006); P. McDonald *et al.*, Astrophys. J. **635**, 761 (2005).
- [45] R. Barbieri, G. R. Dvali and L. J. Hall, Phys. Lett. B **377**, 76 (1996) [arXiv:hep-ph/9512388]; R. Barbieri, L. J. Hall, S. Raby and A. Romanino, Nucl. Phys. B **493**, 3 (1997) [arXiv:hep-ph/9610449].
- [46] L. L. Everett and A. J. Stuart, Phys. Lett. B **698**, 131 (2011) [arXiv:1011.4928 [hep-ph]].